

Recap: Simple modules, composition series and the theorem of Jordan and Hölder

Definition: Let R be a ring. An R -module S is called simple $\Leftrightarrow S \neq \{0\}$, and $\{0\}$ and S are the only submodules of S .

An R -module M is called semisimple $\Leftrightarrow M = \underbrace{S_1 \oplus \dots \oplus S_n}_{\text{a finite direct sum}}$ for some simple modules S_1, \dots, S_n (not necessarily different).

Theorem (Schur's Lemma): Let S be a simple R -module. Then the endomorphism ring $\text{End}_R(S)$ is a division ring; that is, any non-zero endomorphism φ of S is invertible.

Proof: $\ker(\varphi)$ is a submodule of $S \Rightarrow \varphi = 0$ or φ injective
 $\text{im}(\varphi)$ is a submodule of $S \Rightarrow \varphi = 0$ or φ surjective \square

Corollary (Schur's Lemma, special case): Let R be a finite dimensional K -algebra, $K = \bar{K}$ an algebraically closed field, and S a simple R -module. Then $\text{End}_R(S) = \{ \lambda \cdot \text{id} : \lambda \in K \}$.

Proof: Any $a \in S, a \neq 0$, generates $S: R \cdot a = S$ since Ra is a submodule.
 $\Rightarrow S$ is finite dimensional over K . $\Rightarrow \varphi$ has an eigenvalue λ
 $\varphi - \lambda \cdot \text{id}$ is an endomorphism, not invertible $\Rightarrow \varphi = \lambda \cdot \text{id} \square$

Definition: Let R be a ring and M an R -module. A composition series of M is a finite chain of submodules $0 = M_0 \subset M_1 \subset \dots \subset M_{n-1} \subset M_n$ for some $n \in \mathbb{N}$ such that each M_{j+1}/M_j is simple for $j = 0, \dots, n-1$.

(We allow $M = 0, n = 0$.)

(A composition series may not exist and it may not be unique.)

Example: $R = M = \mathbb{Z}, \mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset \dots, \mathbb{Z} \supset 3\mathbb{Z} \supset 9\mathbb{Z} \supset \dots, \mathbb{Z} \supset 2\mathbb{Z} \supset 6\mathbb{Z} \supset 12\mathbb{Z} \supset 36\mathbb{Z} \supset \dots$

are infinite chains with simple subquotients M_{j+1}/M_j . Describe these subquotients

quotient of a submodule

Simple \mathbb{Z} -modules (= simple abelian groups) are, up to isomorphism, of the form $\mathbb{Z}/p\mathbb{Z}$ for p a prime number $\Rightarrow \mathbb{Z}$ cannot have a finite composition series.

If S and T are simple R -modules then $M = S \oplus T$ has the composition series $0 \subset S \subset S \oplus T$ and $0 \subset T \subset S \oplus T$, but the composition factors (= simple subquotients) are the same: S and T .

\Rightarrow We consider two composition series as equivalent iff they have the same length and the same composition factors (counting with multiplicity), but possibly indexed differently. if and only if

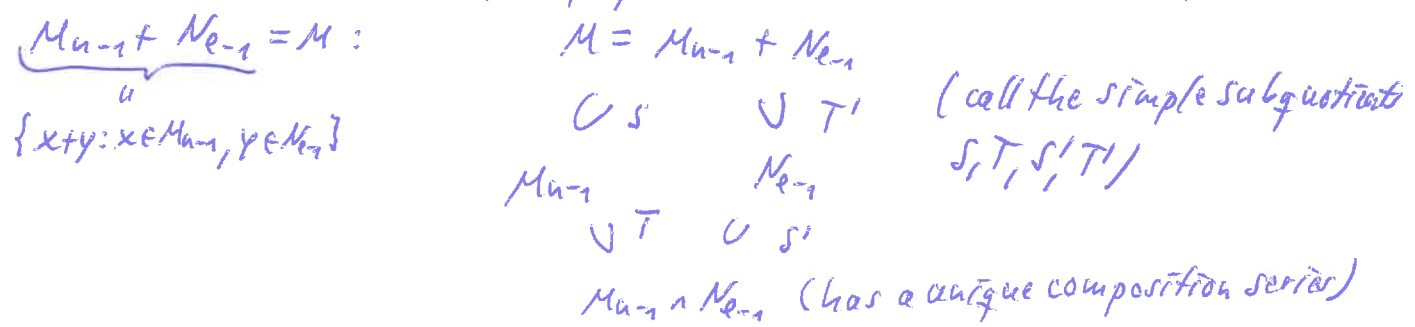
A typical uniqueness result then is:

Theorem (Jordan-Hölder): Let R be a finite dimensional K -algebra and M a finite dimensional R -module. Then M has a unique composition series. (up to equivalence)

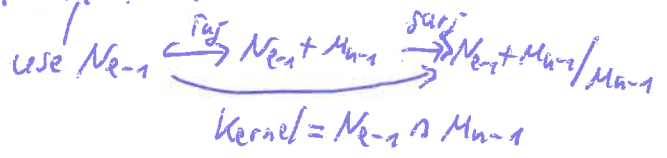
Proof: $M = 0$ or M simple \checkmark Proceed by induction, eg on $\dim_K M$.

Suppose $M = M_n \supset \dots \supset M_1 \supset M_0 = 0$ are two composition series.
 $\Rightarrow N_\ell \supset \dots \supset N_1 \supset N_0 = 0$

If $M_{n-1} = N_{\ell-1}$ then by induction: $n-1 = \ell-1$ and $M_{n-1} = N_{\ell-1}$ has a unique composition series. $M_n / M_{n-1} = M_n / N_{\ell-1}$ is the additional composition factor. Otherwise, $M_{n-1} \cap N_{\ell-1}$ is a proper submodule of M_{n-1} and of $N_{\ell-1}$, and



$S = M / M_{n-1} = \frac{M_{n-1} + N_{\ell-1}}{M_{n-1}} \cong \frac{N_{\ell-1}}{M_{n-1} \cap N_{\ell-1}} = S'$ and similarly $T \cong T'$ \square



This also works for $R = \mathcal{U}$ and M a finite abelian group.

Moreover, there is an analogous Jordan-Hölder theorem (with the same proof) for finite groups (not R -modules for fixed any R) - here one has to be careful to work with normal subgroups to define simple groups and to get quotients that are groups.

Exercise: $R := K[x]$, K a field

- find different "infinite composition series" of the R -module ${}_R R$
- let $f(x) \in K[x]$, $f(x) \neq 0$, let $M := R / \langle f(x) \rangle$
find a composition series of M and ^{find} the composition factors
is this composition series unique?

A composition series can be viewed as a sequence of short exact sequences

$$\begin{aligned} 0 &\rightarrow M_1 \xrightarrow{\text{incl}} M_2 \rightarrow M_2/M_1 \rightarrow 0 \\ 0 &\rightarrow M_2 \xrightarrow{\text{incl}} M_3 \rightarrow M_3/M_2 \rightarrow 0 \\ 0 &\rightarrow M_3 \xrightarrow{\text{incl}} M_4 \rightarrow M_4/M_3 \rightarrow 0 \\ &\dots \end{aligned}$$

where M_1 and all the right hand terms are simple.