Recap: Simple modules, composition series and the theorem of Jordan and Hölder

Definition: Let R be a ring. An R-module S is called simple $\Leftrightarrow$ $S \neq \{0\}$, and $\{0\}$ and S are the only submodules of S.
An R-module M is called semisimple $\Leftrightarrow$ $M = \underbrace{S_1 \oplus \ldots \oplus S_n}_{\text{a finite direct sum}}$ for some simple modules $S_1, \ldots, S_n$ (not necessarily different).

Theorem (Schur's Lemma): Let S be a simple R-module. Then the endomorphism ring $End_R(S)$ is a division ring; that is, any non-zero endomorphism φ of S is invertible.
Proof: $\ker(\varphi)$ is a submodule of S $\Rightarrow$ $\varphi = 0$ or φ injective
$im(\varphi)$ is a submodule of S $\Rightarrow$ $\varphi = 0$ or φ surjective $\square$

Corollary (Schur's Lemma, special case): Let R be a finite dimensional k-algebra, $k = \bar{k}$ an algebraically closed field, and S a simple R-module. Then $End_R(S) = \{\lambda \cdot id : \lambda \in k\}$.
Proof: Any $a \in S$, $a \neq 0$, generates S: $R \cdot a = S$ since Ra is a submodule.
$\Rightarrow$ S is finite dimensional over k. $\overset{k=\bar{k}}{\Rightarrow}$ φ has an eigenvalue λ
$\varphi - \lambda id$ is an endomorphism, not invertible $\Rightarrow$ $\varphi = \lambda \cdot id$ $\square$

Definition: Let R be a ring and M an R-module. A composition series of M is a finite chain of submodules $0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n$ for some $n \in \mathbb{N}$ such that each M_{j+1}/M_j is simple for $j = 0, \ldots, n-1$.
(We allow $M = 0$, $n = 0$.)
(A composition series may not exist and it may not be unique.)
Example: $R = M = \mathbb{Z}$, $\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset \ldots$, $\mathbb{Z} \supset 3\mathbb{Z} \supset 9\mathbb{Z} \supset \ldots$, $\mathbb{Z} \supset 2\mathbb{Z} \supset 6\mathbb{Z} \supset 12\mathbb{Z} \supset 36\mathbb{Z} \ldots$
are infinite chains with simple $\underbrace{\text{subquotients}}_{\text{quotient of a submodule}}$ M_{j+1}/M_j. describe these subquotients

Simple $\mathbb{Z}$-modules (= simple abelian groups) are, up to isomorphism, of the form $\mathbb{Z}/p\mathbb{Z}$ for p a prime number $\Rightarrow$ $\mathbb{Z}$ cannot have a finite composition series.

If S and T are simple R-modules then $M = S \oplus T$ has the composition series $0 \subset \underbrace{S}_{S} \subset \underbrace{S \oplus T}_{T}$ and $0 \subset \underbrace{T}_{T} \subset \underbrace{S \oplus T}_{S}$, but the composition factors (= simple subquotients) are the same: S and T.

sub-quotients

$\Rightarrow$ We consider two composition series as equivalent $\underbrace{\text{iff}}_{\text{if and only if}}$ they have the same length and the same composition factors (counting with multiplicity), but possibly indexed differently.

A typical uniqueness result then is:

<u>Theorem</u> (Jordan–Hölder): Let R be a finite dimensional k-algebra and M a finite dimensional R-module. Then M has a unique composition series.

up to equivalence

Proof: $M = 0$ or M simple ✓ Proceed by induction, eg on $\dim_k M$.

Suppose $M = M_n \supset \dots \supset M_1 \supset M_0 = 0$ are two composition series.

$\qquad = N_\ell \supset \dots \supset N_1 \supset N_0 = 0$

If $M_{n-1} = N_{\ell-1}$ then by induction: $n-1 = \ell-1$ and $M_{n-1} = N_{\ell-1}$ has a unique composition series. $M_n / M_{n-1} = M_n / N_{\ell-1}$ is the additional composition factor.

Otherwise, $M_{n-1} \cap N_{\ell-1}$ is a proper submodule of M_{n-1} and of $N_{\ell-1}$, and

$\underbrace{M_{n-1} + N_{\ell-1}}_{\{x+y:\, x \in M_{n-1},\, y \in N_{\ell-1}\}} = M$:

$$\begin{array}{ccccc} & & M = M_{n-1} + N_{\ell-1} & & \\ & \cup\, S & & \cup\, T' & \\ M_{n-1} & & & & N_{\ell-1} \\ & \cup\, T & & \cup\, S' & \\ & & M_{n-1} \cap N_{\ell-1} & & \end{array}$$

(call the simple subquotients S, T, S', T')

$M_{n-1} \cap N_{\ell-1}$ (has a unique composition series)

$$S = M/M_{n-1} = \frac{M_{n-1} + N_{\ell-1}}{M_{n-1}} \simeq \frac{N_{\ell-1}}{M_{n-1} \cap N_{\ell-1}} = S' \text{ and similarly } T \simeq T' \quad \square$$

use $\underbrace{N_{\ell-1} \xhookrightarrow{\text{inj}} N_{\ell-1} + M_{n-1} \xrightarrow{\text{surj}} N_{\ell-1} + M_{n-1} / M_{n-1}}_{\text{Kernel} = N_{\ell-1} \cap M_{n-1}}$

This also works for $R = \mathbb{Z}$ and M a finite abelian group.
Moreover, there is an analogous Jordan-Hölder theorem (with the same proof) for finite groups (not R-modules for ~~finit~~ any R) - here one has to be careful to work with normal subgroups to define simple groups and to get quotients that are groups.

Exercise: $R := k[x]$, k a field

- find different "infinite composition series" of the R-module ${}_R R$
- let $f(x) \in k[x]$, $f(x) \neq 0$, let $M := R/\langle f(x) \rangle$
  find a composition series of M and find the composition factors
  is this composition series unique?

A composition series can be viewed as a sequence of short exact sequences

$$0 \to M_1 \xrightarrow{\text{incl}} M_2 \to M_2/M_1 \to 0$$
$$0 \to M_2 \xrightarrow{\text{incl}} M_3 \to M_3/M_2 \to 0$$
$$0 \to M_3 \xrightarrow{\text{incl}} M_4 \to M_4/M_3 \to 0$$
$$\cdots$$

where M_1 and all the right hand terms are simple.