Preview: Why are we going to study short exact sequences of modules?

General principle: Try to reduce problems to smaller cases.
Example: R ring, X an R-module. X decomposable $\Leftrightarrow$ $X \simeq X_1 \oplus X_2$, where X_1, X_2 are R-modules, too, both $\neq 0$. If we know X_1 and X_2 well, then we may also know much about X.
(Under some assumptions, X has a unique decomposition $X \simeq X_1 \oplus \dots \oplus X_n$ into indecomposables. See the recap on the theorem of Krull-Remak-Schmidt.)
More generally, X may be the middle term of a short exact sequence of R-modules: $0 \to X_1 \xrightarrow{\alpha} X \xrightarrow{\beta} X_2 \to 0$
(α injective, β surjective, image(α) = kernel(β), see the recap on short exact sequences)
When $X \simeq X_1 \oplus X_2$ choose $\alpha: X_1 \to X$ the inclusion
and $\beta: X \to X_2$ the projection
to get a split exact sequence.
When X has a submodule Y, one can form a short exact sequence (for short: ses)

$$0 \to Y \xrightarrow{\alpha} X \xrightarrow{\beta} X/Y \to 0 \text{ with } \alpha = \text{inclusion}.$$

(When X is simple it has no non-trivial submodule or quotient and thus cannot be the middle term of a non-trivial ses.
Under some assumptions, a general X can be built up from simple modules by iterated ses, producing a composition series, which may be essentially unique. See the recap on composition series and the Jordan-Hölder theorem.)

Problem: How ~~we~~ can we build up larger modules from smaller ones by forming ses with the given modules as end terms?

$$0 \to A \to ? \to C \to 0 \quad (A \text{ and } C \ R\text{-modules})$$

Of course, there always is an ses with $? = A \oplus C$, ie a split ses. But are there other ses that do not split, and if so, how many - or how many really different ones?

Let us look at some examples:

- $R = \mathbb{Z}$. $A = \mathbb{Z}/2\mathbb{Z}$ and $C = \mathbb{Z}/3\mathbb{Z}$ are simple R-modules why?

  There is an ses $0 \to A \xrightarrow{\alpha} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\beta} C \to 0$ define α and β and check ses

  However, this sequence splits and $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ why?

  and there is no non-split ses with end terms A and C (or the other way round). why?

  Keep $R = \mathbb{Z}$ and try $A = C = \mathbb{Z}/3\mathbb{Z}$:

  There is an ses $0 \to A \xrightarrow{\alpha} \mathbb{Z}/9\mathbb{Z} \xrightarrow{\beta} C \to 0$ define α and β and check ses

  and this sequence does not split. why? (hint: orders of elements)

  What is the difference? any conjectures?

- $R = K$, a field. Do you find any non-split short ~~exact~~ exact sequence?

  $R = \mathrm{Mat}(n \times n, K)$, the algebra of $n \times n$-matrices over a field.

  Do you find any non-split ses?

- $R = K[x]$, polynomial ring in one variable over a field

  Recall or check: An R-module M is a K-vectorspace V together with a K-linear endomorphism $x: V \to V$.

  Examples of R-modules: $M = \left(K^2, x: K^2 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} K^2\right)$, that is: M has a K-basis v_1, v_2 and $x: v_1 \mapsto \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$

  $$v_2 \mapsto v_1$$

  $N = (K, x: K \xrightarrow{0} K)$, K-basis v_3, $x: v_3 \mapsto 0$

  There is an ses of R-modules

  $$0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} M/N \to 0$$

  check

  $$\alpha: v_3 \mapsto v_1 \quad \beta: v_1 \mapsto 0$$
  $$v_2 \mapsto \text{basis element of } M/N \simeq N$$

This sequence does not split (why) and the Jordan block $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ does not have an easier block diagonal form.

Keep $R = K[x]$ and $V = K^2$.

Define M_1 by $x : V \xrightarrow{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}} V$

and M_2 by $x : V \xrightarrow{\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}} V$

Is M_1 or M_2 the middle term of a split or non-split short exact sequence of R-modules?

If yes: write it down. If no: why not?

What makes the difference? any conjectures?

- $R = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, upper triangular 2×2-matrices over a field (subalgebra of $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$)

$R \ni e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$Re_1 = \{re_1 : r \in R\}$ and Re_2 are R-submodules of R.

There is an ses of R-modules

$$0 \to Re_1 \xrightarrow{\text{inclusion}} R \longrightarrow R/Re_1 \to 0 \quad \text{and} \quad R/_{Re_1} \cong Re_2$$

This sequence splits. check

There also is an ses of R-modules

$$0 \to Re_1 \to Re_2 \to \overbrace{Re_2/Re_1}^{\text{cokernel} \cong} \to 0$$

which does not split. define α and β and check non-split

Here, $Re_1 \ncong$ cokernel. why not?

does this say anything about your conjectures?

These examples tell us (and are supposed to do so) that the situation is complicated. Given R-modules X and Y we don't know how many (or how many really different) ses $0 \to X \to E \to Y \to 0$ there are. In particular, we do not know how to decide if (X, Y given) always $E \cong X \oplus Y$, or something else can appear as middle term.

(The unknown middle term E often is called an extension of Y by X, since X is isomorphic (via α) to a submodule of E such that the quotient is Y:

$$X \xrightarrow{\alpha} \boxed{\begin{array}{c} \text{"}Y\text{"} \\ \hline \alpha(X) \end{array}} = E.)$$

So, the problem has become bigger:

How to describe ses $0 \to X \to E \to Y \to 0$?

How to describe extensions E of Y by X?

How many are really different? What means really different?

How to decide if all such ses split?

How to decide if always $E \simeq X \oplus Y$?

Approach:
- decide what "really different" means by putting an equivalence relation on the set of ses $0 \to X \to E \to Y \to 0$
- hope for surprises: this ~~ol~~ set has an algebraic structure!
- decide when it is trivial and that this means all ses split
- develop methods to compute this structure, when X and Y are given

Working out this approach will open ~~the~~ a door for us and lead us into a new area, homological algebra – which provides crucial tools not only for representation theory, but also for algebraic topology, algebraic geometry, microlocal analysis, –