§ 9. Morita equivalences

Let $A = k$ be a field and $B = Mat(n \times n, k)$ for some $n \geq 2$. In A-Mod there is up to isomorphism one simple module, k, which has endomorphism ring k. In B-Mod there is up to isomorphism one simple module, $\begin{pmatrix} k \\ \vdots \\ k \end{pmatrix}$, which also has endomorphism ring k. Both A and B are semisimple k-algebras, and they seem to have "the same" representation theory, although they certainly are not isomorphic. What does it mean to have the same representation theory? And how does one know if two rings do have the same representation theory?

9.1 Definition: Two rings A and B are Morita equivalent if and only if the categories A-Mod and B-Mod are equivalent as categories.

This looks reasonable, but other definitions may offer themselves:

A-mod $\simeq$ ~~A~~ B-mod for A, B finite dimensional?

A-Mod $\simeq$ B-Mod as abelian categories?

A-Mod $\simeq$ B-Mod and the equivalence is an additive functor, an exact functor, ..?

Thus we first check if some of these properties follow from definition 9.1.

9.2. ~~Definition~~ Proposition: Let $F: A\text{-Mod} \rightarrow B\text{-Mod}$ be an equivalence of categories. Then (a) F preserves the properties of being epimorphism, monomorphism, isomorphism (hence also surjective, injective, bijective).

(b) F sends kernels to kernels, cokernels to cokernels, images to images, the zero module to the zero module.

(c) F is exact: It maps exact sequences to exact sequences.

(d) F sends products to products, coproducts to coproducts, submodules to submodules, quotient modules to quotient modules, simple modules to simple modules.

(e) F preserves addition: $\forall X,Y$: $\mathrm{Hom}_A(X,Y) \underset{f \mapsto F(f)}{\xrightarrow{F}} \mathrm{Hom}_B(F(X),F(Y))$ is an isomorphism of abelian groups.

(f) F maps projective modules to projective modules, injective modules to injective modules.

(g) F sends finitely generated A-modules to finitely generated B-modules

Proof: Since F is an equivalence, there exists a quasi-inverse functor $G: B\text{-Mod} \to A\text{-Mod}$ and a natural isomorphism $\eta: 1_{A\text{-Mod}} \longrightarrow G\circ F$.
If G really would be the inverse of F, properties like epimorphism etc could be transfered easily between A-Mod and B-Mod, since they can be defined in terms of category theory. In the general situation we have to use η.

(a) Let $f: X \to Y$ be an epimorphism in A-Mod. Claim: $F(f)$ is an epimorphism in B-Mod.

Let $g: F(Y) \to Z$ and $h: F(Y) \to Z$ morphisms such that $g\circ F(f) = h\circ F(f)$.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] & F(Y) \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Z \end{tikzcd}$$

We have to show: $g=h$.

Apply $G \Rightarrow G(g)\circ G\circ F(\overset{f}{g}) = G(h)\circ G\circ F(f)$

precompose η_X $\Rightarrow$ $G(g)\circ G\circ F(\overset{f}{g})\circ \eta_X = G(h)\circ G\circ F(\overset{f}{g})\circ \eta_X$

$\Rightarrow$ $G(g)\circ \eta_Y \circ f = G(h)\circ \eta_Y\circ f$

f epi $\Rightarrow$ $G(g)\circ \eta_Y = G(h)\circ\eta_Y$

η_Y iso $\Rightarrow$ $G(g) = G(h)$

G is faithful $\Rightarrow$ $g = h$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & G\circ F(X) \arrow[d, "(G\circ F)(f)"] \\ Y \arrow[r, "\eta_Y"] & G\circ F(Y) \arrow[d, shift right, "G(g)"'] \arrow[d, shift left, "G(h)"] \\ & G(Z) \end{tikzcd}$$

(η is natural)

The other assertions in (a) follow in a similar way.

(b) also follows in a similar way.

(c) follows from (a) and (b), since exact means equality of certain images and kernels.

(d) follows similarly (products, coproducts) or by applying the previous assertions.

(e) is more of a challenge. We have to define addition of morphisms in a categorical way.

Let $\alpha, \beta \in \mathrm{Hom}_A(X,X)$. We are looking for a categorical definition of $\alpha+\beta$

$X\oplus X$ is both a product and a coproduct (one says: $X\oplus X$ is a biproduct) – this is a special case of the general fact that finite coproducts are products, too, and vice versa

$$\begin{tikzcd}
& X\oplus X \arrow[dl, "\bar{\pi}_1"'] \arrow[dr, "\bar{\pi}_1"] & \\
X & & X \\
& \forall Z \arrow[ul] \arrow[ur] \arrow[uu, dashed, "\exists!"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& X\oplus X \arrow[dd, dashed, "\exists!"'] & \\
X \arrow[ur, "\iota_1"] \arrow[dr] & & X \arrow[ul, "\iota_2"'] \arrow[dl] \\
& \forall Z' &
\end{tikzcd}$$

which one is the product and which one is the coproduct?

We use these properties to define new maps:

$$\begin{tikzcd}
& X\oplus X \arrow[dl, "\bar{\pi}_1"'] \arrow[dr, "\bar{\pi}_2"] & \\
X & & X \\
& X \arrow[ul, "1_X"] \arrow[ur, "1_X"'] \arrow[uu, dashed, "\exists!\Delta"] &
\end{tikzcd}$$

$\Delta: X \to X\oplus X$ (diagonal map) s.that $\bar{\pi}_1\circ\Delta = 1_X$ and $\bar{\pi}_2\circ\Delta = 1_X$

$$\begin{tikzcd}
& X\oplus X \arrow[dd, dashed, "\exists!\nabla"] & \\
X \arrow[ur, "\iota_1"] \arrow[dr, "1_X"'] & & X \arrow[ul, "\iota_2"'] \arrow[dl, "1_X"] \\
& X &
\end{tikzcd}$$

$\nabla: X\oplus X \to X$ (codiagonal) s.that $\nabla\circ\iota_1 = 1_X$ and $\nabla\circ\iota_2 = 1_X$

Now we define (categorically) a "matrix" $h = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}: X\oplus X \longrightarrow Y\oplus Y$

$$\begin{tikzcd}
& X\oplus X \arrow[dd, dashed, "\exists!h_1"] & \\
X \arrow[ur, "\iota_1"] \arrow[dr, "\alpha"'] & & X \arrow[ul, "\iota_2"'] \arrow[dl, "0"] \\
& Y &
\end{tikzcd}$$

$h_1\circ\iota_1 = \alpha$

$h_1\circ\iota_2 = 0$

and

$$\begin{tikzcd}
& X\oplus X \arrow[dd, dashed, "\exists!h_2"] & \\
X \arrow[ur, "\iota_1"] \arrow[dr, "0"'] & & X \arrow[ul, "\iota_2"'] \arrow[dl, "\beta"] \\
& Y &
\end{tikzcd}$$

$h_2\circ\iota_1 = 0$

$h_2\circ\iota_2 = \beta$

(in both cases we use $X\oplus X$ is a coproducts)

Now we use $Y\oplus Y$ is a product.

$$\begin{tikzcd}
& Y\oplus Y \arrow[dl, "\bar{\pi}_1"'] \arrow[dr, "\bar{\pi}_2"] & \\
Y & & Y \\
& X\oplus X \arrow[ul, "h_1"] \arrow[ur, "h_2"'] \arrow[uu, dashed, "\exists!h"] &
\end{tikzcd}$$

This defines $h: X\oplus X \to Y\oplus Y$ and the notation $h = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ is appropriate.

Finally we compose the maps obtained so far:

$$\begin{tikzcd}
X \arrow[r, "\Delta"] & X\oplus X \arrow[r, "h"] & Y\oplus Y \arrow[r, "\nabla"] & Y \\
x_0 \arrow[r, mapsto] \arrow[u, phantom, "\in" rotate=90] & (x_0, x_0) \arrow[r, mapsto] & (\alpha(x_0), \beta(x_0)) \arrow[r, mapsto] & \alpha(x_0)+\beta(x_0)
\end{tikzcd}$$

$\Rightarrow \alpha+\beta = \Delta_Y \circ h \circ \Delta_X$, this defines addition in purely categorical terms, and hence F preserves addition. This implies (e).

(f) follows from the categorical definitions of projective and injective.

(g) is less obvious. Since submodules are preserved, we prove a characterisation of finitely generated modules in terms of chains of submodules.

Claim: A module M is finitely generated $\Leftrightarrow$ $\forall$ totally ordered index set I, totally ordered family of submodules X_i (so. $i<j \Rightarrow X_i \subset X_j$): if ~~b~~ all $X_i \neq M$, then $\bigcup_{i\in I} X_i \neq M$

Proof of the claim: "$\Rightarrow$" Let M be finitely generated with generators $m_1, -, m_n$. If $\bigcup X_i = M$ then $\exists X_{i_1} \ni m_1, X_{i_2} \ni m_2, -, X_{i_n} \ni m_n$. The X_i form a chain (totally ordered by inclusion) $\Rightarrow \exists j: X_{i_j} \supset$ all $X_{i_1}, -, X_{i_n} \Rightarrow X_{i_j} \ni m_1, -, m_n \Rightarrow X_{i_j} = M$

"$\Leftarrow$" We assume M is not finitely generated. To reach a contradiction we use Zorn's lemma to find a chain X_i, all $\neq M$, with $\bigcup X_i = M$.

Let $\mathcal{S} = \{ Y \subseteq M \mid M/Y$ is not finitely generated$\}$. $\mathcal{S} \neq \emptyset$ since $0 \in \mathcal{S}$ by assumption on M. If there exists $Y_0 \in \mathcal{S}$ that is maximal with respect to inclusion, then $Y_0 \subsetneq M$ since M/Y_0 not f.g. $\Rightarrow \exists\, m_0 \in M - Y_0 \Rightarrow Y_0 + Rm_0 \supsetneq Y_0$.
$\llcorner$ we are in R-Mod
$\Rightarrow Y_0 + Rm_0 \notin \mathcal{S} \Rightarrow M/_{(Y_0+Rm_0)}$ is finitely generated, by $\overline{m_1}, -, \overline{m_n}$ (residue classes represented by $m_1, -, m_n$).
$\Rightarrow M/Y_0$ is finitely generated by $\overline{m_0}, \overline{m_1}, -, \overline{m_n}$ ↯

To reach this ~~[illegible]~~ contradiction we have to show that such a maximal Y_0 exists. By Zorn's lemma it does exist provided every totally ordered chain X_i in $\mathcal{S}$ ($\llcorner i \in I$) does have an upper bound in $\mathcal{S}$. $\bigcup_{i\in I} X_i$ is an upper bound. We have to check that $X = \bigcup_{i\in I} X_i \in \mathcal{S}$, i.e. that M/X is not finitely generated. Assume, for a contradiction, it is finitely generated by $\overline{m_1}, -, \overline{m_n}$.

Note that we still have to use the assumption of "$\Leftarrow$". The X_i and X are not exactly what we need, since $X \subsetneq M$ is possible.

However, $\hat{X} := \langle X, m_1, -, m_n \rangle$ (the R-module generated by $X, m_1, -, m_n$) satisfies $\hat{X} = M$ since $M/X = \langle \overline{m_1}, -, \overline{m_n} \rangle$. Therefore, set $\hat{X_i} := \langle X_i, m_1, -, m_n \rangle$. Then $\hat{X} = \bigcup_{i\in I} \hat{X_i}$. And ~~$M/\hat{X_i}$~~ $\hat{X_i} \neq M$ since M/X_i is not finitely generated, in particular not by $\overline{m_1}, -, \overline{m_n}$. So, the $\hat{X_i}$, $i \in I$, form a chain as in the assumption and by assumption $\hat{X} = \bigcup_{i\in I} \hat{X_i} \neq M$ ↯ □

Proposition 9.2 lists many properties and data, which are preserved. What is missing in this list, however, is the algebra A. Actually, an equivalence $A\text{-Mod} \xrightarrow{F} B\text{-Mod}$ need not send A to B.

What do we know about $F(A)$? $F(A) \in B\text{-Mod}$ has to be projective by 9.2(f) and finitely generated by 9.2(g). There is one more property of A that is preserved:

9.3 Definition: A module $G \in A\text{-Mod}$ is a generator of $A\text{-Mod}$ $\Leftrightarrow$ for each $M \in A\text{-Mod}$ there exists an epimorphism $\bigoplus_{i \in I} G \twoheadrightarrow M$ for some index set I (depending on M).

A module $P \in A\text{-Mod}$ is called a progenerator $\Leftrightarrow$ P is finitely generated projective and a generator.

So, the regular module ${}_AA \in A\text{-Mod}$ is an example of a progenerator. By 9.2(e), an equivalence F sends a generator to a generator, hence a progenerator – for instance ${}_AA$ – to a progenerator.

The surprise is: We can use any progenerator to define an equivalence.

What does it mean? If there is an equivalence $A\text{-Mod} \xrightarrow{F} B\text{-Mod}$, then $F(A)$ is a progenerator in $B\text{-Mod}$. This information does not help us to find B. But F has a quasi-inverse $G: B\text{-Mod} \to A\text{-Mod}$, sending B to $G(B)$, a progenerator in $A\text{-Mod}$, which is given.

9.4 Theorem (Morita, 1958): Let R and S be rings. Then the following assertions are equivalent:

(a) R and S are Morita equivalent, i.e. $R\text{-Mod}$ and $S\text{-Mod}$ are equivalent categories.

(b) $\exists$ progenerator $P \in R\text{-Mod}$ such that $S = \mathrm{End}_R(P)^{op}$.

[(c) $\exists$ progenerator $Q \in \text{Mod-}R$ such that $S = \mathrm{End}_R(Q)$.]

(We only prove (a) $\Leftrightarrow$ (b).)

So, R is Morita equivalent to all rings S that are the opposite rings of $\mathrm{End}_R(P)$ for P a progenerator, and R is not Morita equivalent to any other ring.

When R is a finite dimensional k-algebra, then S ~~[illegible]~~ is finite dimensional, too, since P progenerator is finite dimensional and $\mathrm{End}_R(P) \subset \mathrm{End}_k(P)$.
In this case, $R\text{-Mod} \xrightarrow{\sim} S\text{-Mod}$ restricts to an equivalence $R\text{-mod} \xrightarrow{\sim} S\text{-mod}$.
Conversely, if there is an equivalence $F: R\text{-mod} \xrightarrow{\sim} S\text{-mod}$, then $F(R)$ is a progenerator in S-mod and S is Morita equivalent to $\mathrm{End}_S(F(R))^{op} \underset{\substack{\uparrow \\ F \text{ equivalence}}}{\simeq} \mathrm{End}_R(R)^{op} = R$
Hence: $R\text{-Mod} \simeq S\text{-Mod} \Leftrightarrow R\text{-mod} \simeq S\text{-mod}$

When R is a finite dimensional k-algebra, the regular module ${}_RR$ is a direct sum of indecomposable projective modules

$$ {}_RR = P_1^{d_1} \oplus P_2^{d_2} \oplus \dots \oplus P_n^{d_n} $$

where P_1 is the projective cover of $S_1, \dots, P_n$ of S_n ($S_1, \dots, S_n$ simples$/\simeq$).
Then P is a progenerator in R-Mod $\Leftrightarrow P \simeq P_1^{e_1} \oplus \dots \oplus P_n^{e_n}$ for $e_1, \dots, e_n \in \mathbb{N}$ (all are ≥ 1). Thus the algebras Morita equivalent to R are, up to isomorphism, the algebras $\mathrm{End}_R(P_1^{e_1} \oplus \dots \oplus P_n^{e_n})^{op}$.
Therefore we get all rings Morita equivalent to R by iterating the following three operations:
- replace a ring by an isomorphic one: $S \rightsquigarrow S' \simeq S$
- replace a ring S by $\mathrm{Mat}(n \times n, S)$ for some n
- replace a ring S by eSe for an idempotent e with Se being a generator in S-Mod

Proof of Theorem 9.4, (a) $\Leftrightarrow$ (b).
(a) $\Rightarrow$ (b): $S\text{-Mod} \xrightarrow[G]{\sim} R\text{-Mod} \Rightarrow G(S)$ is a progenerator P in R-Mod and and $S^{op} \simeq \mathrm{End}_S(S) \simeq \mathrm{End}_R(G(S)) = \mathrm{End}_R(P)$
(b) $\Rightarrow$ (a): Let ${}_RP \in R\text{-Mod}$ be a progenerator and $\mathrm{End}_R(P)^{op} = S$. We have to define functors $R\text{-Mod} \underset{G}{\overset{F}{\rightleftarrows}} S\text{-Mod}$, which are mutually quasi-inverse equivalences.
By 9.2(c), an equivalence must be an exact functor. Thus a Hom-functor $\mathrm{Hom}_R(X,-)$ with X projective may be a good choice. Since $\mathrm{Hom}_R(X,Y)$ is supposed to be an S-module, X should have an S^{op}-structure on the right $\rightsquigarrow$ choose $X = {}_RP_S$.
(See chapter 3, page 3.2, for the bimodule structure of $\mathrm{Hom}(X,Y)$.)

This defines $F = \mathrm{Hom}_R({}_RP_{\underset{\mathrm{End}_R(P)^{op}}{S^{op}}}, -) : R\text{-Mod} \to S\text{-Mod}$.

We need a quasi-inverse G of F. An equivalence and a quasi-inverse form an adjoint pair and we know a left adjoint of F by Theorem 8.9.

$\rightsquigarrow$ Define $G = {}_RP_S \otimes - : S\text{-Mod} \to R\text{-Mod}$.

We have to show that $F \circ G \simeq \mathrm{id}_{S\text{-Mod}}$ and $G \circ F \simeq \mathrm{id}_{R\text{-Mod}}$

Define a candidate natural isomorphism

$$\varphi : \mathrm{id}_{S\text{-Mod}} \longrightarrow F \circ G \quad \text{for } M \in S\text{-Mod by}$$

$$\varphi_M : M \longrightarrow (F \circ G)(M) = \mathrm{Hom}_R({}_RP_S, {}_RP_S \otimes M)$$

$$m \longmapsto (p \mapsto p \otimes m)$$

Claim: φ is natural in M, ie φ is a natural transformation

Proof: Let $f : M_1 \to M_2$ be a homomorphism

$$\begin{tikzcd} M_1 \arrow[r, "\varphi_{M_1}"] \arrow[d, "f"'] & \mathrm{Hom}_R(P, P \otimes M_1) \arrow[d, "\mathrm{Hom}(1, 1 \otimes f)"] \\ M_2 \arrow[r, "\varphi_{M_2}"'] & \mathrm{Hom}(P, P \otimes M_2) \end{tikzcd} \qquad \begin{tikzcd} m_1 \arrow[r, mapsto] \arrow[d, mapsto] & (p \mapsto p \otimes m_1) \arrow[d, mapsto] \\ f(m_1) \arrow[rd, mapsto] & (p \mapsto p \otimes f(m_1)) \arrow[d, phantom, "\shortparallel"] \\ & (p \mapsto p \otimes f(m_1)) \end{tikzcd} \quad \checkmark$$

Claim: φ_M is an isomorphism for each M

Proof: We proceed in several steps

First case: $M = S$.

We first check $X \underset{S}{\otimes} S \simeq X$ for each X, by showing that X satisfies the universal property of the tensor product:

$$\begin{tikzcd} X \times S \arrow[r, "(x,s) \mapsto xs"] \arrow[rd, "f"'] & X \arrow[d, dashed, "\hat{f}"] \\ & G \arrow[u, phantom, "\wr", near end] \end{tikzcd}$$

where $\hat{f}$ is defined by

$$x_0 \longmapsto f(x_0, 1)$$

check that this works and is natural in X

This implies $\mathrm{Hom}_R(P, P \underset{S}{\otimes} S) \simeq \mathrm{Hom}_R(P, P) \simeq S$ (these are module isomorphisms, not ring isomorphisms).

Similarly $\mathrm{Hom}_R(P, P \underset{S}{\otimes} S^n) \simeq S^n$ for $n \in \mathbb{N}$.

Moreover, for any index set I, $Hom_R(P, P\otimes_S(\bigoplus_{i\in I} S)) \simeq \bigoplus_{i\in I} S$, since P is finitely generated and thus $f: P \to P\otimes_S(\bigoplus_{i\in I} S) \cong \bigoplus_{i\in I} P$ has image in a finite sum of copies of P. This means f is given by
$f': P \to P\otimes_S S^e$ for some ℓ and thus given by $\varphi(x)$ for some $x\in S^\ell \simeq Hom_R(P, P\otimes S^e)$
$\Rightarrow \varphi_{\bigoplus_I S}$ is surjective. It is also injective because an element $x\in\bigoplus_I S$ is a finite linear combination of elements in S^ℓ, some ℓ, and thus again the isomorphism $S^e \simeq Hom_R(P, P\otimes S^e)$ applies (together with naturality of φ).
$\rightsquigarrow$ M = free module works as well, which is the second case.

Third (and general) case: M is any module.
$\Rightarrow \exists$ free presentation $S_2 = \bigoplus_{i\in J} S \xrightarrow{f} S_1 = \bigoplus_{i\in I} S \xrightarrow{g} M \to 0$ (exact)
Since φ is a natural transformation, the following diagram is commutative:

$$\begin{tikzcd}
S_2 \arrow[r, "f"] \arrow[d, "\varphi_{S_2}"'] \arrow[dr, phantom, "\circlearrowright"] & S_1 \arrow[r, "g"] \arrow[d, "\varphi_{S_1}"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, "\varphi_M"'] \arrow[dr, phantom, "\circlearrowright"] & 0 \arrow[d, "0"'] \\
Hom_R(P, P\otimes S_2) \arrow[r] & Hom_R(P, P\otimes S_1) \arrow[r] & Hom_R(P, P\otimes M) \arrow[r] & 0
\end{tikzcd} \quad (*)$$

φ_{S_2} and φ_{S_1} are isomorphisms by the previous case.
By a diagram chase, φ_M is an isomorphism.
$\rightsquigarrow$ $\varphi: id_{S\text{-}Mod} \to F\circ G$ is a natural isomorphism.

We also need a natural isomorphism $\psi: G\circ F \to id_{R\text{-}Mod}$. For N in R-Mod, set $\psi_N: {}_RP_S \otimes_S Hom_R({}_RP_S, {}_RN) \to N$
$p_0 \otimes f \longmapsto f(p_0)$
The proof that ψ is a natural isomorphism proceeds exactly in the same way as for φ:

Naturality: Given $N_1 \xrightarrow{g} N_2$:

$$\begin{tikzcd}
P\otimes_S Hom_R(P, N_1) \arrow[r] \arrow[d, "1_P\otimes Hom(P,g)"] & N_1 \arrow[d, "g"] \\
P\otimes_S Hom_R(P, N_2) \arrow[r] & N_2
\end{tikzcd}
\qquad
\begin{tikzcd}
p_0\otimes f \arrow[r, mapsto] \arrow[d, mapsto] & f(p_0) \arrow[d, mapsto] \\
p_0\otimes g\circ f \arrow[r, mapsto] & \underset{g\circ f(p_0)}{\overset{g(f(p_0))}{\shortparallel}}
\end{tikzcd}$$

Ψ_N is a natural isomorphism is again done in several steps:
First case: $N = P$, then $P \otimes_S \underbrace{\mathrm{Hom}_R(P,N)}_{=S} = P \otimes_S S = P = N$

Then consider $N = \bigoplus_I P$ and use as before that P is finitely generated to get
$P \otimes_S \mathrm{Hom}_R(P, \bigoplus_I P) \simeq \bigoplus_I P = N$
And in the third step we resolve N by P, which is a progenerator and get an analogue of $(*)$:

$$\begin{tikzcd}
P \otimes_S \mathrm{Hom}_R(P,P_2) \arrow[r] \arrow[d, "\wr\ \Psi_{P_2}"] \arrow[dr, phantom, "\circlearrowleft"] & P \otimes \mathrm{Hom}_R(P,P_1) \arrow[r] \arrow[d, "\wr\ \Psi_{P_1}"] \arrow[dr, phantom, "\circlearrowleft"] & P \otimes \mathrm{Hom}_R(P,N) \arrow[r] \arrow[d, "\Psi_N"] & 0 \\
P_2 \arrow[r] & P_1 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

resolution of N by copies of P

$\Rightarrow \Psi_N$ isomorphism

$\square$

Example: Let A be an algebra and ${}_AA = P_1^{d_1} \oplus - \oplus P_n^{d_n}$. Let $P = P_1^{e_1} \oplus - \oplus P_n^{e_n}$ where $\forall i: 1 \leq e_i \leq d_i$. Then ${}_AP$ is a direct summand of ${}_AA$ ~~ad~~ and there exists an idempotent $e = e^2 \in A$ such that $P = Ae$.
What is $\mathrm{End}_A(P)$?
Recall: $\mathrm{End}_A({}_AA) \cong A^{op}$: ~~All~~ A homomorphism $f: {}_AA \to {}_AA$ is right multiplication by $a := f(1)$: $f(x) = f(x \cdot 1) = x f(1) = x a$, and every $a \in A$ defines such a homomorphism, which we call f_a. For $a, b \in A$: $(f_b \circ f_a)(x) = (xa)b =$
$= x(ab) = f_{ab}(x) \Rightarrow f_{ab} = f_b \circ f_a \Rightarrow \mathrm{End}_A({}_AA) \simeq A^{op}$
$P = Ae$: We will show $\mathrm{End}_A(P) = eAe^{op}$
Since e and $(1-e)$ are orthogonal idempotents and $1_A = e + (1-e)$, we can decompose A as abelian group: $A = 1_A \cdot A \cdot 1_A = eAe \oplus (1-e)Ae \oplus eA(1-e) \oplus (1-e)A(1-e)$
or better as a "matrix": $= \begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}$

An endomorphism $f: \overset{\overset{P}{\shortparallel}}{Ae} \to Ae$ can be extended to $\hat{f}: {}_AA \to {}_AA$ by mapping the complemen $A(1-e)$ of Ae to zero.
$\hat{f}$ then is right multiplication by $\hat{a} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}$

For $x \in Ae$ and $y \in A(1-e)$, $\hat{f}$ sends $(x\ y)$ to $(x\ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which must be $(f(x)\ 0)$ by definition of $\hat{f}$, which also implies $b=c=0$, and $d=0$ as well.

$\Rightarrow f: Ae \rightarrow Ae$ is given by ~~$a \in eA$~~ right multiplication with $a \in eAe$.

Since $a \in eAe$, it satisfies $a = eae$.

$\Rightarrow$ Endomorphisms of Ae are given by right multiplication with elements in eAe (and such a right multiplication is, of course, a left module homomorphism).

This is true for any choice of $e = e^2$.

For the particular choice in this example, A and eAe are Morita equivalent.

9.5 Definition: Let A be a finite dimensional k-algebra and ${}_AA = P_1 \oplus \dots \oplus P_n$ with $P_1, \dots, P_n$ indecomposable projective with $P_i \not\cong P_j$ for $i \neq j$. Then A is called a basic algebra.

So, in a decomposition of the regular module of a basic algebra, every isomorphism class of indecomposable projective modules occurs exactly once. Equivalently, $1 = e_1 + \dots + ~~e_n~~\ e_n$ is a sum of pairwise orthogonal primitive idempotents with $Ae_i \not\cong Ae_j$ for $i \neq j$.

9.6 Corollary: Every finite dimensional algebra over a field k is Morita equivalent to a basic k-algebra.

Thus, when attempting to prove a result in representation theory, we are often allowed to work just with basic algebras. (But to determine explicitly the basic algebra of, for instance, a group algebra in finite characteristic, can be a hard problem.

Path algebras (if finite dimensional) and their finite dimensional quotients kQ/I are basic algebras.

Other examples of basic k-algebras are field extensions $L \supset k$: 1_L cannot be written as a non-trivial sum of pairwise orthogonal primitive idempotents. $L \supsetneq k$ cannot be isomorphic to any kQ/I, since, for instance, kQ/I - simples have dimension 1 over k, while L is an L-simple of dimension > 1.

If one doesn't want to work with (skew-) field extensions, one can restrict the class of algebras further:

9.7 Definition: A k-algebra A is <u>elementary</u> if $A/\text{rad}A$ is isomorphic to a product of copies of k. Equivalently, each simple A-module is one-dimensional over k. why is this an equivalent condition?

A basic means: the semisimple k-algebra $A/\text{rad}A$ is a product of 1×1 matrix rings over skew fields. So: elementary $\Rightarrow$ basic.

Now we can justify our choice of examples in chapter 4:

9.8 Theorem: (Gabriel) Let A be a finite dimensional k-algebra. Then A is elementary $\Leftrightarrow$ $A \simeq$ ~~A~~ $kQ/_I$, a bound quiver algebra.

In terms of simple modules, A is elementary $\Leftrightarrow$ each simple module S is one-dimensional over k. By Schur's Lemma, this is equivalent to saying $\text{End}_A(S) \simeq k$. A Morita equivalence from A to eAe is an equivalence of categories and thus sends simple modules to simple modules with the same endomorphism ring.

9.9 Corollary: Let A be a finite dimensional k-algebra such that every simple A-module has endomorphism ring isomorphic to k. Then A is Morita equivalent to a bound quiver algebra $kQ/_I$.

how to prove this?

~~When~~ When K is algebraically closed, the assumption in 9.9 is satisfied.

Proof of 9.8: We will show that there exist a path algebra kQ and a surjective algebra homomorphism $\varphi : kQ \twoheadrightarrow A$ with $I = \mathrm{Ker}(\varphi)$ an admissible ideal.

The first step is to find a suitable quiver Q. When $A \simeq kQ/I$, Corollary 4.8 tells us that Q_0 corresponds to the isomorphism classes of simple A-modules and, less obviously, the number of arrows from a to b ($a, b \in Q_0$) equals the k-dimension of $\mathrm{Ext}^1_A(S(a), S(b))$. This also tells us there is no choice; we must define Q in this way. In terms of quiver representations this corresponds to the minimal projective resolutions of $S(a)$ starting as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus\limits_{\substack{a\to b\\ \text{arrow}}} P(b) \arrow[rr] \arrow[dr] & & P(a) \arrow[r] & S(a) \arrow[r] & 0 \\
& & \mathrm{rad}\, P(a) \arrow[ur] & & &
\end{tikzcd}$$

generated by paths starting at a (when $P(a)$ is a right module)

(For kQ/I the resolution need not stop at the second step.)

We also know that $\mathrm{rad}(kQ/I)$ is generated, as a k-vectorspace, by all paths of length at least one. Hence, as an ideal it is generated by arrows.

So, using this for A we define the quiver Q_A in the same way: $Q_0 = \{1, .., n\}$ corresponds to the isomorphism classes of simple A-modules (right modules), and the arrows $a \to b$ in Q_1 are a k-basis of $\mathrm{Ext}^1_A(S(a), S(b))$ for a, b in Q_0.

Again, there sequences $0 \to \mathrm{rad}\, P(a) \to P(a) \to S(a) \to 0$ and projective resolutions

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus P(b) \arrow[rr] \arrow[dr] & & P(a) \arrow[r] & S(a) \arrow[r] & 0 \\
& & \mathrm{rad}\, P(a) \arrow[ur] & & &
\end{tikzcd}$$

which we choose to be minimal. Applying $\mathrm{Hom}_A(-, S(b))$ yields

$$0 \to \mathrm{Hom}_A(S(a), S(b)) \xrightarrow{\alpha} \mathrm{Hom}_A(P(a), S(b)) \xrightarrow{\beta} \mathrm{Hom}_A(\mathrm{rad}\, P(a), S(b)) \xrightarrow{\mu}$$
$$\to \mathrm{Ext}^1_A(S(a), S(b)) \to 0$$

Here, α is an isomorphism, since each map $S(a) \to S(b)$ lifts to $P(a) \to S(b)$ and each $P(a) \to S(b)$ sends $\mathrm{rad}\, P(a)$ to zero and thus factors through $S(a)$.

This implies $\beta = 0$ and μ is an isomorphism, too.

What are the maps $\mathrm{rad}\, P(a) \to S(b)$?

Let $f: \text{rad}\, P(a) \longrightarrow S(b)$ be a module homomorphism. Then $f(\underbrace{\text{rad}(\text{rad}\, P(a))}_{\text{rad}^2 P(a)}) \cancel{\to} \subset \text{rad}\, S(b) = 0$ and f factors through $\text{rad}\, P(a) \to \text{rad}\, P(a) / \text{rad}^2 P(a)$ which is semisimple.

Each simple summand $\simeq S(b)$ defines a map to $S(b)$ and the others don't. These simple summands are the images of the maps $P(b) = e_b A \longrightarrow \text{rad}\, P(a)/\text{rad}^2 P(a)$, and these images add up to $(\text{rad}\, P(a)/\text{rad}^2 P(a))\, e_b$.

Since $P(a) = e_a A$ and $\text{rad}\, P(a) = e_a \cdot \text{rad}(A)$, this yields

$$\text{Ext}^1_A(S(a), S(b)) \simeq e_a \cdot \frac{\text{rad}(A)}{\text{rad}^2(A)} \cdot e_b$$

Thus, each arrow $a \xrightarrow{\alpha} b$ has a preimage in $e_a\, \text{rad}(A) \cdot e_b$ representing the equivalence class of $a \xrightarrow{\alpha} b$ in $e_a \frac{\text{rad}(A)}{\text{rad}^2(A)} e_b$.

Replacing α ~~the~~ by this preimage (and calling that α from now on), we realise the arrows α as elements of $\text{rad}(A)$.

(Note that for $kQ/_I$ we used the inverse bijection: An arrow α was an element in $kQ/_I$ and was shown to correspond to a basis element of $\text{Ext}^1_A(S(a), S(b))$.)

From now on: the arrows $\alpha: a \to b$ are elements of $e_a\, \text{rad}(A)\, e_b$.

Claim 1: The arrows in Q_1 generate $\text{rad}(A)$ as an ideal.

Proof of claim 1: The residue classes of the arrows form a k-basis of $\frac{\text{rad}(A)}{\text{rad}^2(A)}$.

Let $x_0 \in \text{rad}(A)$. Then $x_0 = (\text{linear combination of arrows}) + x_1$, for $x_1 \in \text{rad}^2(A)$.

$\text{rad}^2(A) = \text{rad}(A) \cdot \text{rad}(A) \Rightarrow x_1$ is a sum of $y_1 \cdot z_1$ with $y_1, z_1 \in \text{rad}(A)$.

y_1 and z_1 are linear combinations of arrows plus "error terms" $y_2, z_2 \in \text{rad}^2(A)$.

$\Rightarrow x_1$ is a linear combination of products (with two factors) of arrows (i.e. paths of lengths two) plus an "error term" $x_2 \in \text{rad}^3(A)$.

Replacing x_1 in the sum writing x_0 by the linear combination of products plus x_2 we get $x_0 = (\text{sthing generated by arrows}) + x_2$, where the error term x_2 now is in $\text{rad}^3(A)$.

Writing x_2 as linear combinations of paths of length three plus x_3 we get a new error term in $\mathrm{rad}^4(A)$, and so on.
But $\mathrm{rad}(A)$ is nilpotent: $\mathrm{rad}^n(A) = 0$ for $n \gg 0$ $\Rightarrow$ In the n-th step, the error term vanishes, and Claim 1 follows.

Claim 2: There exists a surjective algebra homomorphism

$$\varphi : kQ_A \twoheadrightarrow A$$

Proof of claim 2: $Q_0 = \{1, \dots, n\}$ and $A/\mathrm{rad}A \cong \prod_{i=1}^{n} k$ or, in other words, $1_A = e_1 + \dots + e_n$ (sum of primitive pairwise orthogonal idempotents) and $A/\mathrm{rad}A = ke_1 \oplus ke_2 \oplus \dots \oplus ke_n$. Thus we set $\varphi(\underset{\substack{\uparrow \\ \text{vertex}}}{i}) := \underset{\substack{\uparrow \\ \text{idempotent}}}{e_i}$

and, of course, $\varphi(\underset{\substack{\text{arrow} \\ \text{in } Q_1}}{\alpha : a \to b}) := \underset{\substack{\text{element} \\ \text{in rad}(A)\text{, more precisely in } e_a \mathrm{rad}(A) e_b}}{\alpha : a \to b}$

~~$\varphi(\alpha_1$~~

and $\varphi(\underbrace{\alpha_1 \cdots \alpha_\ell}_{\text{path of length } \ell \geq 2}) := \alpha_1 \cdots \alpha_\ell$ (product of elements in $\mathrm{rad}(A)$)

and then extend it k-linearly.
This defines a morphism of algebras (there are no relations in kQ_A).

$$\left.\begin{aligned} \varphi : Q_0 &\longrightarrow \text{basis of } A/\mathrm{rad}A \\ Q_1 &\longrightarrow \text{generators of rad} A \end{aligned}\right\} \Rightarrow \varphi \text{ is surjective.} \quad \square$$

We are not yet done, however: In the definition of bound quiver algebra $kQ/_I$ it is required that I is an admissible ideal (Definition 4.7): $\exists n \geq 2$ such that $(kQ)_{\geq n} \subseteq I \subseteq (kQ)_{\geq 2}$.

Claim 3: ~~The~~ $I := \mathrm{Ker}(\varphi)$ is an admissible ideal in kQ_A.
Proof of claim 3: By definition of φ, it sends

$$kQ_0 \xrightarrow{\sim} \prod_{i=1}^{n} ke_i$$

and $(kQ)_{\geq 1}$ to $\mathrm{rad}(A)$

This implies $(kQ)_{\geq \ell} \xrightarrow{\varphi} \mathrm{rad}^{\ell}(A)$

Since A is finite dimensional $\mathrm{rad}^{\ell}(A)=0$ for $\ell \gg 0$.

$\Rightarrow (kQ)_{\geq \ell} \subseteq I$ for $\ell \gg 0$.

We know already that $I \subseteq (kQ)_{\geq 1}$, but we have to show that $I \subseteq (kQ)_{\geq 2}$. φ maps $(kQ)_{\geq 2}$ into $\mathrm{rad}^2(A)$ and $(kQ)_1$ isomorphically to $\mathrm{rad}(A)/\mathrm{rad}^2(A)$, by definition of $Q=Q_A$. $\Rightarrow (kQ_{\leq 1}) \cap \mathrm{Ker}\,\varphi = \{0\}$ and we are done with Claim 3, and with the whole proof. □

Finally, let us have a look at the functors used to prove Theorem 9.4: Hom-functors and tensor product functors. In the situation we just considered, there is a third functor, which is naturally isomorphic to a Hom- and to a tensor product functor:

Let $e=e^2 \in R$ (a ring). Multiplication by e on the left defines a functor

$$e\cdot - : R\text{-Mod} \to eRe\text{-Mod}$$

$$M \mapsto eM = \{em : m \in M\}$$

$$(f: M \to N) \mapsto (f|_{eM} : eM \to eN)$$

$$em \mapsto f(em) = ef(m)$$

check this is a functor

Then: $eR \underset{R}{\otimes} - : R\text{-Mod} \to eRe\text{-Mod}$, $e\cdot - : R\text{-Mod} \to eRe\text{-Mod}$ and $\mathrm{Hom}_R(Re,-) : R\text{-Mod} \to eRe\text{-Mod}$ are naturally isomorphic to each other. What are the natural isomorphisms?

Up to natural isomorphism, Hom and tensor functors are, however, determined by properties that we have used in the context of Morita's theorem:

9.10 Theorem (Theorem of Eilenberg and Watts): Let R and S be rings and $F: R\text{-Mod} \to S\text{-Mod}$ an additive functor. Then the following two assertions are equivalent:

(a) $\exists\, {}_S M_R$, a bimodule, such that $F \simeq {}_S M_R \otimes -$

(b) F is right exact and preserves direct sums (that is, $F(\bigoplus_{i \in I} M_i) \cong \bigoplus_{i \in I} F(M_i)$)

There are also versions of this theorem for left exact functors, characterising Hom-functors. A contravariant left exact functor converting direct sums into direct products is naturally isomorphic to a $\mathrm{Hom}(-, M)$ for some M. (See the exercises on natural isomorphisms for the condition that direct sums get converted into direct products.) For the covariant Hom functor $\mathrm{Hom}(M,-)$ the characterising conditions are to be left exact and to preserve direct products.

One can show that the conditions just discussed are equivalent to having a right adjoint (in the case of ~~left~~ right exact functors) ~~or any~~ or a left adjoint (in the case of left exact functors).

Proof of 9.10:

(a) ⇒ (b): By 8.2, the tensor product is right exact, since it is a left adjoint. To get isomorphisms $M \otimes (\underbrace{\bigoplus_{i\in I} X_i) \simeq \bigoplus_{i\in I} M \otimes X_i}$, one can check that $M \otimes (\bigoplus_{i\in I} X_i)$ is the coproduct of the $M \otimes X_i$; coproducts are unique up to isomorphism, even up to unique isomorphism.

(b) ⇒ (a): First we have to find the bimodule M. When F is the tensor product $M \underset{R}{\otimes} -$, $F(R) \simeq M$. So, for the given F as in (b), we define $M := F(R)$. M is a left S-module, since $F: R\text{-Mod} \to S\text{-Mod}$.

F induces a map $\mathrm{Hom}_R(R,R) \longrightarrow \mathrm{Hom}_S(F(R), F(R)) = \mathrm{Hom}_S(M, M)$.

$\mathrm{Hom}_R(R,R)$ is a right R-module (using the bimodule structure on $\mathrm{Hom}(X,Y)$ we often have used): For $a \in R$, let $f_a : R \to R$ (right multiplication by a), and $f \cdot f_a : 1 \mapsto a \mapsto f(a) = af(1)$ $\Big[\, x1 = x \overset{1 \mapsto a}{\longmapsto} xa$

$\overset{\cap}{\mathrm{Hom}_R(R,R)}$ $\Rightarrow f \cdot f_a \cdot f_b : 1 \mapsto b \mapsto ba \mapsto f(ba) = baf(1)$

$= f \cdot f_{ba}$ ⇒ right module

Then $F(f_a)$ acts on $F(R)$ and $F(R)$ becomes a right R-module.

So: ${}_S M_R = F(R)$ is a bimodule, since the left S-action and the right R-action commute: for $x \in F(R)$, $\underset{\overset{\cap}{S}}{s}(xa) = s(F(f_a)(x)) = F(f_a)(sx) = (sx)a$

We have to prove that F is naturally isomorphic to $M \underset{R}{\otimes} -$. This requires isomorphisms $M \underset{R}{\otimes} X \simeq F(X)$ for $X \in R\text{-Mod}$. We define these using the universal property of the tensor product:

$$\begin{aligned} \text{Let } \alpha_X :\ & M \times X \longrightarrow F(X) \\ & (m, x_0) \longmapsto F(f_{x_0})(m) \quad \text{where } F(f_{x_0}) : \overset{\overset{M}{\shortparallel}}{F(R)} \longrightarrow F(X) \text{ and} \end{aligned}$$

$$\begin{aligned} f_{x_0} :\ & R \rightarrow X \\ & a \mapsto a x_0, \text{ ie } 1 \mapsto x_0 \end{aligned}$$

(for $X = R$, f_{x_0} is defined as above, so here we generalise the above definition)

To get a map $M \underset{R}{\otimes} X \xrightarrow{\eta_X} F(X)$ we have to check (according to Definition 8.6) that α_X is R-biadditive. The additivity conditions follow from M and X being modules and F being additive. We have to check the third condition in 8.5:

Let $a \in R$, then α_X sends $(ma, x_0) \longmapsto F(f_{x_0})(ma) = F(f_{x_0})(F(f_a)(m))$

and $(m, a x_0) \longmapsto F(f_{a x_0})(m) = F(f_{x_0})(F(f_a(m)))$,

$$\begin{array}{l} \text{since } f_{a x_0} = f_{x_0} \circ f_a : R \rightarrow R \longrightarrow X \\ \qquad\qquad\qquad\qquad\quad 1 \longmapsto a \longmapsto a x_0 \\ \qquad\qquad\qquad\qquad\qquad\quad 1 \longmapsto x_0 \end{array}$$

$\Rightarrow$ The universal property of the tensor product yields a homomorphism

$$\eta_X : M \otimes X \longrightarrow F(X)$$

which we have to show to be an isomorphism, and natural.

To check that η_X is an isomorphism, we proceed exactly as in the proof of 9.4, since both $M \otimes -$ and F ~~have~~ are right exact and commute with direct sums.

$$\left.\begin{array}{l} X = R:\ M \underset{R}{\otimes} R \xrightarrow{\sim} F(R) = M \ \text{ by definition of } \eta_X \\[2ex] X = \text{projective: use direct sums} \\[2ex] X \text{ arbitrary: use a projective presentation} \end{array}\right\} \text{This works when } \eta_X \text{ is natural.}$$

This uses η being a natural transformation, which we have to check anyway. Let $f : X \rightarrow Y$ be an R-module homomorphism. To show:

$$\begin{tikzcd} \overset{\overset{M}{\shortparallel}}{F(R)} \otimes X \arrow[r, "\mu_X"] \arrow[d, "\mathrm{id} \otimes f"'] & F(X) \arrow[d, "F(f)"] \\ F(R) \otimes Y \arrow[r, "\mu_Y"] & F(Y) \end{tikzcd}$$

is commutative.

9.18

$$\begin{tikzcd}
\mathcal{F}(R) \arrow[d] & X \arrow[dl] & \\
m\otimes x_0 \arrow[rr, mapsto] \arrow[dd, mapsto] & & \mathcal{F}(f_{x_0})(m) \arrow[d, mapsto] \\
& & \mathcal{F}(f)(\mathcal{F}(f_{x_0})(m)) \\
m\otimes f(x_0) \arrow[rr, mapsto] & & \mathcal{F}(f_{f(x_0)})(m)
\end{tikzcd}$$

} Want equality here.

We don't know $\mathcal{F}$, but it sends of course two maps that are equal to the same map. Therefore, we check equality in the domain of $\mathcal{F}$:

$M = \mathcal{F}(R)$, so let $a \in R$ and apply the respective maps:

$$\left.\begin{aligned} f_{f(x_0)}(r) &= rf(x_0) \\ (f\circ f_{x_0})(r) &= f(rx_0) = rf(x_0) \end{aligned}\right\} \Rightarrow f\circ f_{x_0} = f_{f(x_0)}$$

$$\Rightarrow \mathcal{F}(f)\circ\mathcal{F}(f_{x_0}) = \mathcal{F}(f_{f(x_0)})$$

This proves naturality and finishes the proof that $\eta = (\eta_X)_X$ is a natural isomorphism $M\otimes - \xrightarrow{\sim} \mathcal{F}$. $\square$