§8. Tensor product and Tor

In §6 we have constructed Ext^i as right derived functors of Hom. The theory of universal (co) homological δ-functors established in §6 works for two kinds of functors – left exact functors like Hom and right exact functors. The tensor product to be introduced now is the canonical example of a right exact functor and its left derived functors, called Tor, are analogues of Ext.

Before we define the tensor product, we clarify the categorical setup in which Hom and tensor product are supposed to be related.

8.1 Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $L: \mathcal{A} \to \mathcal{B}$ and $R: \mathcal{B} \to \mathcal{A}$ functors. The ordered pair (L, R) is said to be an adjoint pair $:\Leftrightarrow$
$\exists$ natural isomorphisms $\mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow[\tau_{A,B}]{\sim} \mathrm{Hom}_{\mathcal{A}}(A, R(B))$ for all A in $\mathcal{A}$, B in $\mathcal{B}$.
L is called left adjoint (of R or of the pair (L, R)) and R is called right adjoint.

"Natural" isomorphism means (as expected) that for $f: A \to A'$ in $\mathcal{A}$ and $g: B \to B'$ in $\mathcal{B}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) \arrow[r, "L(f)^*"] \arrow[d, "\tau_{A',B}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\tau_{A,B}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \arrow[d, "\tau_{A,B'}"] \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \arrow[r, "R(g)_*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B'))
\end{tikzcd}$$

(The difference to the natural transformations we have seen before is that we have two variables instead of one.)

Example of an adjoint pair: Let $L: R\text{-mod} \to \mathcal{A}b$ be the forgetful functor that sends an R-module to itself, seen as an abelian group, and an R-module homomorphism to itself, seen as a homomorphism of abelian groups.

L has right adjoint $\mathrm{Hom}_{Ab}(R,-)$: For an abelian group B, $\mathrm{Hom}_{Ab}(R,B)$ is a left R-module, since R is a right R-module. The natural isomorphisms required in 8.1 are:

$$\underset{A}{\mathrm{Hom}_{Ab}(\underset{\shortparallel}{L(A)},B)} \xrightarrow[\tau]{\sim} \mathrm{Hom}_{R\text{-mod}}(A,\mathrm{Hom}_{Ab}(R,B)) \quad \text{for } A\in R\text{-mod}, B\in Ab$$

$$f: A\to B \longmapsto (\tau f): a\longmapsto (r\longmapsto f(ra))$$

and

$$a\longmapsto g(a)(1) \longleftarrow g: A\to \mathrm{Hom}_{Ab}(R,B)$$

Check that this works and is natural.

Adjoint pairs of additive functors between abelian categories have a crucial property: {preserve abelian group structure on morphisms}

8.2 Proposition: Let $L:\mathcal{A}\to\mathcal{B}$ and $R:\mathcal{B}\to\mathcal{A}$ be an adjoint pair of additive functors between abelian categories. Then L is right exact and R is left exact.

Hence, left adjoints have left derived functors, according to chapter 6, and right adjoints have right derived functors, assuming that $\mathcal{A}$ has enough projectives and $\mathcal{B}$ has enough injectives.
The existence of left or right adjoints is not guaranteed in general, but needs assumptions on the categories and on the given functor (look for instance for Freyd's "general adjoint functor theorem"), but it is not hopeless to look for adjoints.

Proof of 8.2: Let A be an object in $\mathcal{A}$ and $0\to B'\xrightarrow{\alpha}B\xrightarrow{\beta}B''\to 0$ a short exact sequence in $\mathcal{B}$. $\mathrm{Hom}_{\mathcal{B}}(L(A),-)$ is left exact and by naturality, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(L(A),B') \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{B}}(L(A),B) \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{B}}(L(A),B'') \arrow[d, "\wr"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A,R(B')) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A,R(B)) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A,R(B''))
\end{tikzcd}$$

is commutative with exact rows.

We would like to get that

$$(*) \quad 0 \to R(B') \xrightarrow{R(\alpha)} R(B) \xrightarrow{R(\beta)} R(B'')$$

is exact. When $\mathcal{A} = \Lambda\text{-mod}$ for a ring Λ, we can choose $A := \Lambda$ and get exactness of $(*)$. The general case follows from Yoneda's Lemma, which is discussed below.

$\Rightarrow$ R is left exact (and we only needed to know that R is the right adjoint of some functor L, which we don't know, apart from its existence).

To see that L is right exact, we move to the opposite categories of $\mathcal{A}$ and $\mathcal{B}$, where morphisms go in the opposite direction, but objects are the same.

So, $L^{op}: \mathcal{A}^{op} \to \mathcal{B}^{op}$ does the same as L on objects on morphisms, but these are composed in the opposite direction; $\mathrm{Hom}_{\mathcal{A}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, X)$. Therefore, L^{op} is a right adjoint to R^{op}, hence it is left exact, which implies that L is right exact. $\square$

This proof has already used the following result, which is extremely useful in many situations:

8.3 Theorem (Yoneda's Lemma): Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}\,\mathcal{C}$.

(a) Let $F: \mathcal{C} \to \mathrm{Set}$ be a covariant functor. Then there is a bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set})}(\mathrm{Hom}_{\mathcal{C}}(X, -), F) \xrightarrow{\sim} F(X)$$

$$\eta: \mathrm{Hom}_{\mathcal{C}}(X, -) \to F \longmapsto \eta_X(1_X)$$

(b) Let $G: \mathcal{C} \to \mathrm{Set}$ be a contravariant functor. Then there is a bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Set})}(\mathrm{Hom}_{\mathcal{C}}(-, X), G) \xrightarrow{\sim} G(X)$$

$$\eta: \mathrm{Hom}_{\mathcal{C}}(-, X) \to G \longmapsto \eta_X(1_X)$$

Here, $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ is the category of covariant functors. The morphisms in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ are the natural transformations. The objects in ~~Hom~~ $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ are the contravariant functors from $\mathcal{C}$ to Set.

When $\eta: \mathrm{Hom}_{\mathcal{C}}(X,-) \to F$ is a natural transformation between the functors, it comes with natural maps $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to F(Y)$ for all Y. The statement in (a) is that the natural transformations ~~[illegible]~~ from $\mathrm{Hom}_{\mathcal{C}}(X,-)$ to the functor F are in bijection with the elements of the set $F(X)$. In other words, the one element $\eta_X(1_X)$, where $1_X = \mathrm{id}: X \to X$, determines already the whole natural transformation (and for each element in $F(X)$ there exists a unique natural transformation).

How does 8.3 help to show that (*) in the proof of 8.2 is exact?

We are allowed to freely choose A in the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, R(B')) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B''))$$

Choosing $A = R(B')$ and $\mathrm{id}_A \in \mathrm{Hom}(A = R(B'), R(B'))$: we get that

$$\mathrm{id}_A \longmapsto R(\alpha)\circ \mathrm{id}_A \longmapsto R(\beta)\circ R(\alpha)\circ \mathrm{id}_A \text{ must vanish, hence } R(\beta)\circ R(\alpha) = 0$$

Choosing $A = \mathrm{Ker}(R(\alpha))$ and i = inclusion of $\mathrm{Ker}(R(\alpha))$ into $R(B')$, $R(\alpha)\circ i = 0$ implies i is in the image of $0 \to \mathrm{Hom}_{\mathcal{A}}(A, R(B'))$, which is possible only for $\mathrm{Ker}(R(\alpha)) = 0$.

Choosing $A = \mathrm{Ker}(R(\beta))$ and $j: \mathrm{Ker}(R(\beta)) \to R(B)$ the inclusion, we can write $j = R(\alpha)(\sigma)$ for some $\sigma: \mathrm{Ker}(R(\beta)) \to R(B')$, hence $\mathrm{Im}(j) \subset \mathrm{Im}(R(\alpha))$ and combined with $R(\beta)\circ R(\alpha) = 0$ it follows that $\mathrm{Im}(R(\alpha)) = \mathrm{Ker}(R(\beta))$.

Altogether, this shows (*) is exact.

Proof of 8.3: The two parts are similar and we only check (a).

A natural transformation $\eta: \mathrm{Hom}_{\mathcal{C}}(X,-) \to F$ is a family of morphisms

$$\eta_Y : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to F(Y), \text{ for each object } Y \in \mathrm{Ob}\,\mathcal{C}$$

such that in particular for each morphism $f: X \to Y$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(X,f)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \arrow[d, "F(f)"] \\ \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r] & F(Y) \end{tikzcd}$$

where

$$\begin{tikzcd} 1_X \arrow[r, mapsto] \arrow[d, mapsto] & \eta_X(1_X) \arrow[d, mapsto] \\ f \arrow[r, mapsto] & \eta_Y(f) = F(f)(\eta_X(1_X)) \end{tikzcd}$$

This tells us that $\eta_Y(f)$ is determined by $\eta_X(1_X)$, since F is fixed.
$\Rightarrow \eta \longmapsto \eta_X(1_X)$ is an injective map.
To prove surjectivity we have to choose an arbitrary element $\xi \in F(X)$ and find a suitable natural transformation η with $\xi = \eta_X(1_X)$.
The previous diagram forces us to define $\eta_Y(f) := F(f)(\xi)$ for $f: X \to Y$. But we still have to show that this η is a natural transformation. What is left to be checked is that for any $g: Y \to Z$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\eta_Y"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(g)"'] & F(Y) \arrow[d, "F(g)"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[r] & F(Z)
\end{tikzcd}$$

is commutative,

which for $f: X \to Y$ means:

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \eta_Y(f) = F(f)(\xi) \text{ by definition of } \eta \arrow[d, mapsto] \\
 & F(g)(F(f)(\xi)) \\
g \circ f \arrow[r, mapsto] & \eta_Z(g\circ f) = F(g\circ f)(\xi) \text{ by definition of } \eta
\end{tikzcd}$$

// since F is a functor

$\square$

From Yoneda's Lemma we can derive:

<u>8.4 Corollary:</u> Adjoint functors are unique, up to natural isomorphism.
Proof: Let R and R' be right adjoint to L, then there are natural isomorphisms

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[r, "\sim", "\tau"'] \arrow[dr, "\wr", "\tau'"'] & \mathrm{Hom}_{\mathcal{A}}(A, RB) \arrow[d, "\wr", "\tau' \circ \tau^{-1}"'] \\
 & \mathrm{Hom}_{\mathcal{A}}(A, R'B)
\end{tikzcd}$$

$\Rightarrow$ There are natural isomorphisms $\mathrm{Hom}_{\mathcal{A}}(-, RB) \rightleftarrows \mathrm{Hom}_{\mathcal{A}}(-, R'B)$.
By 8.3 (b),

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})}(\mathrm{Hom}_{\mathcal{A}}(-, RB), \mathrm{Hom}_{\mathcal{A}}(-, R'B)) \simeq \mathrm{Hom}_{\mathcal{A}}(RB, R'B)$$

which implies $RB \simeq R'B$, again natural in B now.
$\Rightarrow R \simeq R'$ as functors. Check the details.
The proof for left adjoints is similar. $\square$

At this point you may protest: 8.3 is for functors $\mathcal{C} \to Sets$, not for functors $\mathcal{A} \to \mathcal{B}$. However, the proof of 8.3 works as well for functors with additional structures, eg additive functors to categories like Ab or k-$Vect$. Since $F(X)$ must be a ~~set~~ set to choose ξ from this set, abelian target categories $\mathcal{B}$ should be embedded in module categories when using 8.3.

Now our task has become clearer: We have to define a functor that is the left adjoint of Hom.

8.5 Definition: Let R be a ring, $A_R \in$ Mod-R and ${}_RB \in R$-Mod and G an abelian group. A function $f: A \times B \to G$ is called R-biadditive $:\Leftrightarrow$ $\forall a, a' \in A, b, b' \in B, r \in R$: $f(a+a', b) = f(a,b) + f(a',b)$

$$f(a, b+b') = f(a,b) + f(a,b')$$
$$f(a, rb) = f(ar, b)$$

If R is commutative and G an R-module and f satisfies in addition $f(ar, b) = f(a, rb) = rf(a,b)$, then f is called R-bilinear.

Examples: • multiplication $R \times R \to R$ is biadditive and when R is commutative it is bilinear

• R commutative, M, N R-modules, then evaluation: $ev: M \times \mathrm{Hom}_R(M,N) \to N$, $(m,f) \mapsto f(m)$ is R-bilinear, where $\mathrm{Hom}_R(M,N)$ is an R-bimodule by $rf: m \mapsto f(rm)$

Special case: $V \times V^* \to k$ for V a k-space with dual V^*.

The tensor product now gets defined as solution to a universal problem:

8.6 Definition: R ring, A_R and ${}_RB$ modules. An abelian group $A \otimes_R B$ is called the tensor product of A and B over R $\Leftrightarrow$ there is a fixed R-biadditive function $h: A \times B \to A \otimes_R B$ such that for every abelian group G and every R-biadditive $f: A \times B \to G$ $\exists!$ $\mathbb{Z}$-homomorphism $\tilde{f}: A \otimes_R B \to G$ making the following diagram commute

$$\begin{tikzcd} A \times B \arrow[r, "h"] \arrow[dr, "f"'] & A \otimes_R B \arrow[d, dashed, "\exists! \tilde{f}"] \\ & G \end{tikzcd}$$

(The map h is part of the structure, thus more precisely one should say the tensor product is $A \otimes B$ together with $h: A \times B \to A \otimes_R B$.)
The uniqueness of $\tilde{f}$ makes the tensorproduct unique, if it exists. Hence it makes sense to call it "the" tensorproduct.

Construction of the tensor product of A_R and $_R B$:
A and B are abelian groups, but we choose a free abelian group F with basis $A \times B$, that is, F has basis (a,b), ordered pairs with $a \in A$, $b \in B$. Let U be the subgroup of F generated by all elements

$$(a, b+b') - (a,b) - (a,b'),\ (a+a', b) - (a,b) - (a',b),\ (ar, b) - (a, rb)$$

Let $A \otimes_R B := F/U$ and denote the coset $(a,b) + U$ by $a \otimes b$.
Define $h: A \times B \to A \otimes_R B$ by $h: (a,b) \mapsto (a,b) + U = a \otimes b$. This is the quotient map $F \to F/U$ restricted to the domain $A \times B \subset F$.
In $A \otimes_R B$ there are equalities (of cosets)

$$\left.\begin{aligned} a \otimes (b+b') &= a \otimes b + a \otimes b' \\ (a+a') \otimes b &= a \otimes b + a' \otimes b \\ ar \otimes b &= a \otimes rb \end{aligned}\right\} \Rightarrow h \text{ is } R\text{-biadditive}$$

Now we check the universal property: Given an abelian group G, $f: A \times B \to G$ biadditive and $i: A \times B \to F$ the inclusion, then:

$$\begin{tikzcd} A \times B \arrow[rr, "h"] \arrow[dr, "i"] \arrow[ddr, "f"'] & & A \otimes_R B \arrow[ddl, dashed, "\hat{f}"] \\ & F \arrow[ur, "proj"] \arrow[d, dashed, "\exists! \varphi"] & \\ & G & \end{tikzcd}$$

F is free abelian with basis $A \times B$
$\Rightarrow \exists! \varphi: F \to G$ extending f, ie
$\varphi(a,b) = f(a,b)\ \forall (a,b) \in A \times B$.
f is R-biadditive $\Rightarrow U \subset \ker(\varphi)$
$\Rightarrow \varphi$ induces $\hat{f}: A \otimes_R B = F/U \longrightarrow G$

where $\hat{f}(a \otimes b) = \hat{f}((a,b)+U) = \varphi(a,b) = f(a,b)$
$\Rightarrow \hat{f} \circ h = f$, so the diagram commutes as required. $\hat{f}$ is unique, since $A \otimes_R B = F/U$ is generated by the residue classes of the basis elements (a,b).
Hence $\underbrace{A \otimes_R B}_{\text{"A tensor B over R"}}$ exists.

By construction, $A \otimes_R B$ is an abelian group (or a vectorspace), but nothing more. When A is a bimodule ${}_S A_R$ then ${}_S A_R \otimes_R {}_R B$ is a left S-module, and $A_R \otimes_R {}_R B_T$ is a right T-module, when ${}_R B_T$ is a right T-module.

Suppose A_R has generators $a_1, \dots, a_n$ (as right ~~A~~-R-module) and ${}_R B$ has generators $b_1, \dots, b_\ell$. Then by construction, an element in $A \otimes_R B$ is a sum of elements of the form $\underbrace{\sum_{i=1}^{n} a_i r_i \otimes \sum_{j=1}^{\ell} s_j b_j}_{u}$ with $r_i, s_j \in R$. By the condition $ar \otimes b = a \otimes rb$

we can rewrite this sum as $\sum_{i,j} a_i r_i s_j \otimes b_j$. But in general, an element $a_1 \otimes b_1 + a_2 \otimes b_2$, for instance, cannot be written as $a \otimes b$, for an $a \in A, b \in B$. $\{a \otimes_R b \mid a \in A, b \in B\}$ in general is a proper subset of $A \otimes_R B$, which consists of finite sums of such elements.

The relation $ar \otimes b = a \otimes rb$ tells us that an expression as a tensor is not unique. And such an element $a \otimes b$ can be zero without a or b being zero.

Example: $\underbrace{\mathbb{Z}/_{2\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/_{3\mathbb{Z}}}_{=0 \text{ (check)}}$: $\bar{1} \otimes \bar{1} = \bar{1} \otimes 2 \cdot \bar{2} = \bar{1} \cdot 2 \otimes \bar{2} = \bar{0} \otimes \bar{2} = \bar{0} \cdot 0 \otimes \bar{2}$
$= \bar{0} \otimes 0 \bar{2}$
$= \bar{0} \otimes \bar{0}$

for elements we often omit R under the tensor sign

The non-uniqueness in writing elements as tensors prevents us from defining maps $A \otimes_R B \to A' \otimes_R B'$ by sending $x \otimes y$ to somewhere. It is, however, possible, to define tensors of maps:

<u>8.7 Proposition:</u> Let $f: A_R \to A'_R$ and $g: {}_R B \to {}_R B'$ be morphisms of right or left A-modules, respectively. Then there is a unique map of abelian groups

$$f \otimes g : A \otimes_R B \longrightarrow A' \otimes_R B' \text{ sending } a \otimes b \text{ to } f(a) \otimes g(b).$$

Proof: We define a map $A \times B \longrightarrow A' \otimes_R B'$ that is R-biadditive, such that by definition of the tensor product this yields a unique $\mathbb{Z}$-homomorphism $A \otimes_R B \longrightarrow A' \otimes_R B'$. Let $\varphi: A \times B \to A' \otimes_R B'$ be $(a,b) \longmapsto f(a) \otimes g(b)$, this is R-biadditive by definition of $A' \otimes_R B'$, for instance

$(ar, b) \longmapsto f(ar) \otimes g(b) = f(a) r \otimes g(b) = f(a) \otimes r g(b) = f(a) \otimes g(rb) \longleftarrow (a, rb)$

$\Rightarrow \exists!$ $\mathbb{Z}$-morphism $A\otimes B \xrightarrow{\hat{\varphi}} A'\underset{R}{\otimes} B'$ and $\hat{\varphi}(a\otimes b) = \varphi(a\times b) = f(a)\otimes g(b)$. □

When A is an S-R-bimodule and A' is so, too, then $f\otimes g$ is a left S-module homomorphism, by the above construction.
Defining maps between tensorproducts by first defining R-biadditive maps generally is a good and safe method.
When more maps are given: $A \xrightarrow{f} A' \xrightarrow{f'} A''$ and $B \xrightarrow{g} B' \xrightarrow{g'} B''$, then composition works as expected: $(f'\otimes g')\circ(f\otimes g) = (f'\circ f)\otimes(g'\circ g)$.
check

8.8 Proposition: $A_R\otimes -$ is a functor $R\text{-Mod} \xrightarrow{F_A} Ab$
and $-\otimes_R B$ is a functor $RG.Mod\text{-}R \xrightarrow{G_B} Ab$.
${}_S A_R \otimes -$ is a functor $R\text{-Mod} \xrightarrow{F_A} S\text{-Mod}$
$-\otimes_R B_T$ is a functor $\text{Mod-}R \xrightarrow{G_B} \text{Mod-}T$.

Proof: Since $1_{A\otimes B} = 1_A\otimes 1_B$, the identity is preserved. Composition is preserved, too: $g: B\to B' \longmapsto 1_A\otimes g$, $g': B'\to B'' \longmapsto 1_A\otimes g'$

$$\Rightarrow F_A(g'\circ g) = 1_A\otimes(g'\circ g) = (1_A\otimes g')(1_A\otimes g) = F_A(g')\circ F_A(g) \;\square$$

F_A and G_B are additive functors: $g,h: B\to B' \Rightarrow 1_A\otimes(g+h) = 1_A\otimes g + 1_A\otimes h$
Since

$$\begin{tikzcd}
a\otimes b \arrow[r, mapsto, "1\otimes(g+h)"] \arrow[dr, mapsto, "1\otimes g + 1\otimes h"'] & a\otimes(g+h)(b) \arrow[d, phantom, "\shortparallel"] \\
& a\otimes g(b) + a\otimes h(b)
\end{tikzcd}$$

Consequence: If $f: A\to A'$ and $g: B\to B'$ are isomorphism, then $f\otimes g: A\otimes B \to A'\otimes B'$ is an isomorphism, too. why?

8.9 Theorem (Adjoint isomorphisms):
(a) Let R and S rings, A_R and ${}_R B_S$ and C_S (bi)modules. Then there is an isomorphism, natural in each variable:

$$\tau_{A,B,C}: \operatorname{Hom}_S(A\underset{R}{\otimes} B_S, C_S) \xrightarrow{\sim} \operatorname{Hom}_R(A_R, \operatorname{Hom}_S({}_R B_S, C_S))$$
$$(f: A\otimes B\to C) \longmapsto \tau(f): a\longmapsto (b\longmapsto f(a\otimes b))$$

So, there are natural isomorphisms

$$\mathrm{Hom}_S(- \otimes_R B, C) \xrightarrow{\sim} \mathrm{Hom}_R(-, \mathrm{Hom}_S(B,C))$$
$$\mathrm{Hom}_S(A \otimes_R -, C) \xrightarrow{\sim} \mathrm{Hom}_R(A, \mathrm{Hom}_S(-,C))$$
$$\mathrm{Hom}_S(A \otimes_R B, -) \xrightarrow{\sim} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,-))$$

(b) Let R and S rings, ${}_RA$ and ${}_SB_R$ and ${}_SC$ (bi) modules. There is an isomorphism, natural in each variable:

$$\tau'_{A,B,C}: \mathrm{Hom}_S({}_SB \otimes_R A, {}_SC) \xrightarrow{\sim} \mathrm{Hom}_R({}_RA, \mathrm{Hom}_S({}_SB_R, {}_SC))$$
$$(f: B \otimes A \to C) \longmapsto \tau'(f): a \mapsto (b \mapsto f(b \otimes a))$$

(Again there are natural isomorphisms in each of the three variables.)

In both cases, one of the factors in the tensor product moves to the right and becomes the domain of a Hom.
The maps τ and τ' may look complicated at first, but they are actually the only reasonable maps to define in these situations.

All the natural isomorphisms say:

$$\boxed{\text{Tensor functors are left adjoint to Hom functors}}$$

8.10 Corollary: Tensor functors are right exact and have left derived functors.

Proof of 8.9 (part (a) only): $\tau_{A,B,C}$ is a homomorphism of abelian groups:

$$(f_1+f_2: A \otimes B \to C) \to \underbrace{\tau(f_1+f_2)}_{\tau(f_1)+\tau(f_2)}: a \mapsto (b \mapsto \underbrace{(f_1+f_2)(a \otimes b)}_{=f_1(a\otimes b)+f_2(a\otimes b)}) \Leftarrow$$

$\tau_{A,B,C}$ is injective: Let $\tau(f)=0$. Then $\forall a \in A, b \in B: f(a \otimes b)=0$. Hence $f=0$.

$\tau_{A,B,C}$ is surjective: Let $g: A_R \to \mathrm{Hom}_{RS}(B,C)$ be an R-homomorphism
$$a \mapsto g(a): B \to C$$
We want $g=\tau(f)$ for some $f: A \otimes B \to C$. So we want: $g(a): b \mapsto f(a \otimes b)$, that is, $f: a \otimes b \mapsto g(a)(b)$. Why does such an f exist? We can at least define $\varphi: A \times B \to C$ Check that φ is R-biadditive:
$$(a,b) \mapsto g(a)(b)$$

$(a, b+b') \longmapsto g(a)(b+b') = g(a)(b) + g(a)(b')$ ✓

$(a+a', b) \longmapsto g(a+a')(b) = g(a)(b) + g(a')(b)$ ✓

$(ar, b) \longmapsto g(ar)(b) = (g(a)\cdot r)(b) = g(a)(rb)$

$(a, rb) \longmapsto g(a)(rb)$ $\quad = \quad$ ✓

$$\Rightarrow \begin{tikzcd} A \times B \arrow[r, "h"] \arrow[dr, "\varphi"'] & A \otimes_R B \arrow[d, dashed, "\exists!\,\tilde\varphi"] \\ & C \end{tikzcd}$$

$\tilde\varphi : A \otimes_R B \to C$ exists and does what we want. Set $f := \tilde\varphi$ and get $\tau(f) = g$.

$\Rightarrow \tau_{A,B,C}$ is an isomorphism.

Let us check one of the naturality conditions: Let $A \xrightarrow{g} A'$ be a morphism

$$\begin{tikzcd} Hom(A \otimes B, C) \arrow[r] & Hom(A, Hom(B,C)) \\ Hom(A' \otimes B, C) \arrow[r] \arrow[u, "(g\otimes 1_B)^*"] & Hom(A', Hom(B,C)) \arrow[u, "g^*"'] \end{tikzcd}$$

$$\begin{tikzcd} f\circ(g\otimes 1_B) : A\otimes B \to C \arrow[r, mapsto] & a \mapsto \tau(\overset{f\circ(g\otimes 1_B)}{\gamma}) : b \mapsto (f\circ(g\otimes 1_B))(a\otimes b) = f(g(a)\otimes b) \\ f : A'\otimes B \to C \arrow[r, mapsto] \arrow[u, mapsto] & a' \mapsto \tau(f) : b \mapsto f(a'\otimes b) \arrow[u, mapsto, "a \mapsto g^*(\tau(f))"'] \end{tikzcd}$$

// ✓

where $g^*(\tau(f)) : b \mapsto f(g(a)\otimes b)$

The other naturality conditions are equally straightforward, and the proof of (b) is very similar. $\square$

So, tensor product is right exact and has left derived functors, which deserve a name:

8.11 Definition: Let R be a ring, A_R and ${}_R B$ R-modules. Then the left derived functors of $A \otimes_R -$ are denoted by $Tor_n^R(A, -)$ for $n \geq 0$.
In particular, $Tor_0^R(A, -) = A \otimes_R -$.

$Tor_n^R(A, B)$ can be computed by choosing a projective resolution P_* of B, forming the chain complex $A \otimes_R P_*$ and then taking homology:

$$Tor_n^R(A, B) = H_n(A \otimes_R P_*)$$

Tor stands for torsion. Soon, we will see why Tor may be called torsion group.

From the general properties of δ-functors we get long exact sequences:
Let $0 \to B' \to B \to B'' \to 0$ be exact. Then there is a long exact Tor-sequence in the second variable:

$$\dots \to Tor_n^R(A,B') \to Tor_n^R(A,B) \to Tor_n^R(A,B'') \to Tor_{n-1}^R(A,B') \to \dots$$
$$\dots \to Tor_1^R(A,B'') \to A \otimes_R B' \to A \otimes_R B \to A \otimes_R B'' \to 0$$

Of course, one can do the same in the first variable and gets the same result:

$$Tor_n^R(A,B) = L_n(A \otimes_R -)(B) = L_n(- \otimes B)(A).$$

Let $0 \to A' \to A \to A'' \to 0$ be exact $\rightsquigarrow$ long exact Tor-sequence in the first variable: $\dots \to Tor_n^R(A',B) \to Tor_n^R(A,B) \to Tor_n^R(A'',B) \to Tor_{n-1}^R(A',B) \to \dots$

$$\dots \to Tor_1^R(A'',B) \to A' \otimes_R B \to A \otimes_R B \to A'' \otimes_R B \to 0$$

Now let us look at abelian groups, where the name Tor comes from:
Let $A = \mathbb{Z}/n\mathbb{Z}$, a $\mathbb{Z}$-module, and B any $\mathbb{Z}$-module. Then A has projective resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. So $P_* = 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0$. Apply $- \otimes_{\mathbb{Z}} B$ and get $0 \to B \xrightarrow{n} \underset{\overset{\shortparallel}{\mathbb{Z} \otimes_{\mathbb{Z}} B}}{B} \to 0$ which has homology $Tor_0^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = B/nB$, $Tor_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = \underbrace{\{b \in B : nb = 0\}}_{\text{"n-torsion elements"}}$, $Tor_\ell^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = 0$ for $\ell \geq 2$

An abelian group is called a torsion abelian group $\Leftrightarrow$ every element has finite order. Finitely generated abelian groups are of the form

$$A = \underbrace{\mathbb{Z}^m}_{\text{torsion-free}} \oplus \underbrace{\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_k\mathbb{Z}}_{\text{torsion}}$$

$\mathbb{Z}^m$ is free and has itself as projective resolution $\rightsquigarrow Tor_0^{\mathbb{Z}}(\mathbb{Z}^m, B) = B$ and $Tor_\ell^{\mathbb{Z}}(\mathbb{Z}^m, B) = 0$ for $\ell \geq 1$.

$\Rightarrow$ In all cases $Tor_1^{\mathbb{Z}}(A,B)$ is a torsion abelian group. This is also true for arbitrary abelian groups A, but we are not going to prove it.

So, over the integers, Tor finds torsion parts of abelian groups. This explains the word, but is not relevant in general situations.

Projective modules are characterised by $Hom_R(P,-)$ being exact or by $Ext^1_R(P,-)=0$ or by $Ext^n_R(P,-)=0$ $\forall n\geq 0$.

8.12 Definition: A left R-module B is flat $\Leftrightarrow$ $-\otimes_R B$ is exact. A right R-module A is flat $\Leftrightarrow$ $A\otimes_R -$ is exact.

This is equivalent to vanishing of $Tor_1^R(A,-)$ or $Tor_1^R(-,B)$, respectively, and also to vanishing of all $Tor_n^R(A,-)$ or $Tor_n^R(-,B)$ for $n\geq 1$.

8.13 Proposition: Projective modules are flat.

Proof: When A (or B) is projective, its projective resolution is just $P_*: \_ 0\to A\to 0 -$ and computing $Tor_1^R(A,X)$ from it, gives values 0 in all degrees $\neq 0$. Similarly for B. $\square$

There are, however, flat modules that are not projective. An example is $\mathbb{Q}$ as a $\mathbb{Z}$-module. For us, the difference between projective and flat is less interesting:

8.14 Proposition: Let A be a finite dimensional k-algebra and P a right A-module. Then $Hom_A(P,-)$ is exact on mod-A $\Leftrightarrow$ $P\otimes_A -$ is exact on A-mod.

Proof: $P\otimes_A -$ exact on A-mod means: $\forall$ ses $0\to X\to Y\to Z\to 0$ in A-mod: $0\to P\otimes_R X\to P\otimes_A Y\to P\otimes_A Z\to 0$ is exact $\Leftrightarrow$ $0\to (P\otimes Z)^*\to (P\otimes Y)^*\to (P\otimes X)^*\to 0$ is exact (where $^* = Hom_k(-,k)$, ie taking dual spaces) $\Leftrightarrow$

$0\to Hom_A(P,Z^*)\to Hom_A(P,Y^*)\to Hom_A(P,X^*)\to 0$ is exact $\Leftrightarrow$ $Hom_A(P,-)$ is exact, since ses $0\to X\to Y\to Z\to 0$ in A-mod correspond bijectively to ses $0\to Z^*\to Y^*\to X^*\to 0$ (here we use that we are considering finite dimensional modules). $\square$

8.9 (adjoint isomorphisms)