# Exercises on tensor products

(1)(a) Let $k$ be a field, $a, b \in \mathbb{N}$. Show that the tensor product $k^a \underset{k}{\otimes} k^b$ is isomorphic to $k^{a \cdot b}$.

(b) The Kronecker product of two matrices $A$ and $B$ is defined as

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1r} & \cdots & a_{1n}b_{11} & \cdots & a_{1n}b_{1r} \\ \vdots & & & & & & & \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & & & & & \\ \vdots & & & & & & & \end{pmatrix}$$

Which operation on linear maps corresponds to the Kronecker product?

(2)(a) Let $k$ be a field, $A$ and $B$ $k$-algebras. Define a $k$-algebra structure on $A \underset{k}{\otimes} B =: C$, such that $A$ and $B$ are isomorphic to subalgebras of $C$.

(b) Let $C$ be a $k$-algebra and $A, B$ subalgebras such that $ab = ba$ $\forall a \in A, b \in B$ and $A \cdot B = C$ and $\dim_k C = \dim_k A \cdot \dim_k B$. Prove that $C \simeq A \underset{k}{\otimes} B$.

(c) Let $G_1, G_2$ be groups. Show that the group algebra over $k$ of $G_1 \times G_2$ is isomorphic to $kG_1 \underset{k}{\otimes} kG_2$, but not in general to $kG_1 \oplus kG_2$.

(Updated February 19th, 2021. Assumptions added in (2)(b).)