Exercises on Yoneda's Lemma

(1) Let $\mathcal{C}$ be a category, $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ the category of contravariant functors $\mathcal{C} \to \mathrm{Set}$, $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ the category of covariant functors $\mathcal{C} \to \mathrm{Set}$. Show that $X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$ defines a full and faithful embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$
and $X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, -)$ — " — $\mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$.

(2) Let G be a finite group and $\mathcal{C}$ a category with just one object $*$ and $\mathrm{Hom}_{\mathcal{C}}(*, *) = G$. Show that $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ are the left G-sets (sets with left G-action) and $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ are the right G-~~action~~ sets.
(a) Which G-sets correspond to $\mathrm{Hom}_{\mathcal{C}}(*, -)$ and to $\mathrm{Hom}_{\mathcal{C}}(-, *)$?
(b) Use (1) to prove Cayley's theorem: G is isomorphic to a subgroup of $\Sigma_{|G|}$, the symmetric group on $|G|$ elements.

(3) Let K be a field and A a K-algebra. Let $\mathcal{C}$ be the category with one object $*$ and $\mathrm{Hom}_{\mathcal{C}}(*, *) = A$. Interpret functors $\mathcal{C} \to K\text{-Vect}$ or $\mathcal{C}^{op} \to K\text{-Vect}$ as A-modules. Which familiar statements are given by Yoneda's Lemma in this situation?

(4) Let K be a field and Q a quiver with KQ finite dimensional. View Q as a category with objects $Q_0 = \{1, \dots, n\}$ and paths as morphisms. What are functors $Q \to K\text{-Vect}$? And what does Yoneda's Lemma say in this situation?