Exercises on adjoint functors

(1) Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ be an adjoint pair of functors between abelian categories.

(a) Suppose F is exact and I is injective in $\mathcal{D}$. Show that $G(I)$ is injective in $\mathcal{C}$. ("G preserves injectives")
Suppose G is exact and P is projective in $\mathcal{C}$. Show that $F(P)$ is projective in $\mathcal{D}$. ("F preserves projectives")

(b) Let R be a ring and I an injective abelian group. Show that $\mathrm{Hom}_{\mathbb{Z}\text{-Mod}}(R, I)$ is an injective R-module.

(c) Let M be an R-module and $I(M) := \prod_{X} \mathrm{Hom}_{\mathbb{Z}\text{-Mod}}(R, \mathbb{Q}/\mathbb{Z})$, where the index set X equals the set of homomorphisms $\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}\text{-Mod}}(R, \mathbb{Q}/\mathbb{Z}))$. Show that there is a monomorphism $M \to I(M)$ and $I(M)$ is injective.
(Hence R-Mod has enough injectives and every R-module has an injective resolution.)

(2) (a) Let $F: \mathcal{C} \to \mathcal{D}$ be an equivalence of categories with quasi-inverse $G: \mathcal{D} \to \mathcal{C}$. Show that (F, G) form an adjoint pair.

(b) Let K be a field and $G: K\text{-Vect} \to \mathrm{Set}$ the forgetful functor sending a vector space V to the set V. Is there a left adjoint F to G?

(c) Let R be a ring, $\mathcal{D} = R\text{-Mod}$ and $\mathcal{C}$ the category of morphisms in R-Mod, that is: objects of $\mathcal{C}$ are morphisms $f: X \to Y$ in R-Mod and morphisms in $\mathcal{C}$ are commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

Show that there are functors $F: \mathcal{D} \to \mathcal{C}$, $M \mapsto (M \xrightarrow{0} 0)$ and $G: \mathcal{C} \to \mathcal{D}$, $(X \xrightarrow{f} Y) \mapsto \mathrm{Ker}(f)$

and that F is left adjoint to G.