§7. Homological dimensions

In this chapter, A is a finite dimensional k-algebra. A-modules are finite dimensional left A-modules.

Recall from 3.12: The projective dimension $\mathrm{pdim}(M)$ of an A-module M is the smallest $n \in \mathbb{N}_0$ such that M has a projective resolution

$$0 \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0$$

When no such M exists, we say $\mathrm{pdim}(M) = \infty$.

$\mathrm{pdim}(M) = 0$ is equivalent to M being projective.

We can split the resolution into short exact sequences

$$0 \to \Omega_1(M) \to P_0 \to M \to 0$$
$$0 \to \Omega_2(M) \to P_1 \to \Omega_1(M) \to 0 \qquad (\text{when } \mathrm{pdim}(M) = n)$$
$$\dots$$
$$0 \to \underset{\overset{\|}{P_n}}{\Omega_n(M)} \to P_{n-1} \to \Omega_{n-1}(M) \to 0$$

and $\Omega_\ell(M) = 0$ for $\ell > n$.

Now we apply dimension shift (Proposition 6.19) to this situation, for an A-module X: $\mathrm{Ext}_A^m(M,X) \simeq \mathrm{Ext}^{m-1}(\Omega_1 M, X) \simeq \mathrm{Ext}_A^{m-2}(\Omega_2 M, X) \simeq \dots$

We can choose m, but the degrees of the Ext-groups have to be at least 1 for the isomorphisms to make sense.

$\Omega_n(M) = P_n$ is projective $\Rightarrow$ $\mathrm{Ext}_A^1(\Omega_n(M), X) = 0 \;\; \forall X$

$$\mathrm{Ext}_A^1(\Omega_n(M), X) \cong \mathrm{Ext}_A^{n+1}(M,X)$$

and for $\ell > n$: $\mathrm{Ext}_A^1(\underbrace{\Omega_\ell(M)}_{=0}, X) = 0 \;\; \forall X$

$$\mathrm{Ext}_A^1(\Omega_\ell(M), X) \cong \mathrm{Ext}_A^{1+\ell}(M,X)$$

$\Rightarrow \mathrm{Ext}_A^{\ell+1}(M,X) = 0 \;\; \forall \ell \geq n \;\; \forall X$

In other words: $\mathrm{Ext}_A^\ell(M,-)=0$ for $\ell > \mathrm{pdim}(M)=n$

(for M projective we know this already)

What about $\mathrm{Ext}_A^n(M,-)$ or Ext in lower degrees?

$\mathrm{Ext}_A^n(M,X) \simeq \mathrm{Ext}_A^1(\Omega_{n-1}(M),X)$, so we should look for an X such that there exists a non-split ses $0\to X\to Y\to \Omega_{n-1}(M)\to 0$

On the previous page there is such a ses

$$0\to\Omega_n(M)\to P_{n-1}\to\Omega_{n-1}(M)\to 0$$

If it would split, $\Omega_{n-1}(M)$ would be a direct summand of the middle term P_{n-1}. This would imply $\Omega_{n-1}(M)$ projective and there would be a projective resolution $0\to\Omega_{n-1}(M)\to P_{n-2}\to \dots\to P_1\to P_0\to M\to 0$ of length $n-1$, which contradicts the minimality of n.

7.1 Corollary: For an A-module M,

$\mathrm{pdim}(M)=n \iff \mathrm{Ext}_A^n(M,-)\neq 0$ and $\forall \ell>n: \mathrm{Ext}_A^\ell(M,-)=0$

$\mathrm{pdim}(M)=\infty \iff \forall n\in\mathbb{N}_0\ \exists \ell>n: \mathrm{Ext}_A^\ell(M,-)\neq 0$

$\iff \forall n\in\mathbb{N}_0: \mathrm{Ext}_A^n(M,-)\neq 0$

use the ses in the projective resolution

There is, of course, an analogous result about injective dimension and injective resolution.

7.2 Definition: The *global dimension* of the algebra A is

$$\mathrm{gldim}(A) := \sup\{\mathrm{pdim}(M) : M\in A\text{-mod}\}\in\mathbb{N}_0\cup\{\infty\}$$

This doesn't look practical, since one has to check all modules, or at least all indecomposable modules.

why?

However, in our situation there is a huge simplification possible:

7.3 Exercise: Let $S_1, \dots, S_n$ be the simple A-modules, up to isomorphism. Prove that
$$\mathrm{gldim}(A) = \max\{\mathrm{pdim}(S_j) : j = 1, \dots, n\}$$

One can define global dimension also for right modules or with respect to injective dimension. One can show that all these values coincide. For general rings this fails.

7.4 Exercise: (a) Show that $\mathrm{gldim}(A) = 0$ is equivalent to A being semisimple.

(b) Show that $\mathrm{gldim}(kQ) \leq 1$ for any finite dimensional path algebra.

(c) Let Q be the quiver $1 \to 2 \to \dots \to n$,

$A_1 := kQ/_{\langle 1\to 2\to \dots \to n\rangle}$ (ie the longest path becomes zero)

$A_2 := kQ/_{kQ_{\geq 2}}$ (ie all paths of length at least two become zero).

Determine $\mathrm{gldim}(A_1)$ and $\mathrm{gldim}(A_2)$.

(d) Suppose that $A\text{-proj} = A\text{-inj}$, that is, $_AA$ is injective. Show that $\mathrm{gldim}(A) \in \{0, \infty\}$ and both cases can occur.

(The algebras A in (d) are called <u>self-injective</u>. Group algebras kG of finite groups G are self-injective, over any field k.)

Let $A = kQ/I$ be a bound quiver algebra and assume that Q contains a loop $\underset{i}{\circlearrowleft}$ at vertex i. For instance, $A = k[x]/_{(x^3)}$ is of this form.

Then the projective resolution of the simple module $S(i)$ has the form

$$\begin{tikzcd}
\dots \arrow[r] & P(i)\oplus P \arrow[rr] \arrow[dr] & & P(i) \arrow[r] & S(i) \arrow[r] & 0 \\
& & \mathrm{rad}\, P(i) = \Omega_1(S(i)) \arrow[ur] & & &
\end{tikzcd}$$

since the loop $\underset{i}{\circlearrowleft}$ is a generator of $\mathrm{rad}\, P(i)$ $(= P(i)_{\geq 1})$.

Surprisingly, this implies infinite global dimension:

Let A be a finite dimensional algebra.

<u>No loop conjecture:</u> If $\mathrm{gldim}(A)<\infty$ then $\mathrm{Ext}^1_A(S,S)=0$ for all simple A-modules. (Proven by Lenzing, 1969, and Igusa, 1990)

<u>Strong no loop conjecture:</u> If $\mathrm{Ext}^1_A(S,S)\neq 0$, then $\mathrm{pdim}(S)=\infty$ (and hence $\mathrm{gldim}(A)=\infty$). (Proven by Igusa, Liu and Paquette, 2011.)

<u>Extreme no loop conjecture:</u> If $\mathrm{Ext}^1_A(S,S)\neq 0$, then $\mathrm{Ext}^i_A(S,S)\neq 0$ for infinitely many i. (Open problem.)

The converse is not true. Example: $Q = 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$, $A = kQ/Q_{\geq 2}$ $(\alpha\beta=0=\beta\alpha)$ is self-injective. check

So, the existence of a loop is sufficient, but not necessary for infinite global dimension.

There is also a numerical criterion for infinite global dimension:
Assume ${}_AA = P_1 \oplus \dots \oplus P_n$, $P_i \not\cong P_j$ for $i\neq j$, all indecomposable, and let $S_1,\dots,S_n$ be the simple quotients of $P_1,\dots,P_n$, respectively. Let $[P_i:S_j]$ be the composition multiplicity of S_j in a composition series (Jordan-Hölder series) of P_i.

<u>7.5 Definition:</u> The <u>Cartan matrix</u> C_A has in the i-th column the vector

$$\begin{matrix}[P_i:S_1]\\ [P_i:S_2]\\ \vdots \\ [P_i:S_n]\end{matrix}$$

When $A = kQ/I$ and P_i is given by a representation of Q, these numbers (composition multiplicities) are just the dimensions of the vector spaces at $1,2,\dots,n$, i.e. the i-th column vector is the dimension vector $\underline{\dim}\, P_i$.

<u>7.6 Exercise:</u> Assume $\mathrm{gldim}(A)<\infty$. Show that $C_A \in \mathrm{Mat}(n\times n,\mathbb{Z})$ is invertible (over $\mathbb{Z}$) and $\det(C_A)=\pm 1$.
i.e. $C_A^{-1} \in \mathrm{Mat}(n\times n, \mathbb{Z})$

(The converse is wrong.)

Cartan determinant conjecture: $\mathrm{gldim}(A) < \infty \Rightarrow \det(C_A) = 1$

(Open problem.)

7.7 Exercise. Let kQ be a finite dimensional path algebra, ie Q has no loops and no oriented cycles. Let $A = kQ/_I$ be a bound quiver algebra wrt this Q. Show that $\mathrm{gldim}(A) < \infty$ and $\det(C_A) = 1$.

7.3 shows that $S := S_1 \oplus \dots \oplus S_n$ (direct sum of all simples) is a test object for global dimension. If we can compute $\mathrm{pdim}(S) = \max\{\mathrm{pdim}(S_i)\}$ then we know $\mathrm{gldim}(A)$ without having to pay attention to other modules. (Note that only countably many indecomposable modules can occur as direct summands of a syzygy of S, hence in general uncountably many indecomposable modules are not involved in computing $\mathrm{pdim}(S)$.)

Is there a test object for projectivity? Let X be an A-module, but we don't know if X is a direct summand of A^n for some n. But we may know that $\mathrm{Ext}^i_A(X,Y) = 0$ in many examples, so X is a suspect of being projective. If X is projective, then $\mathrm{Ext}^i_A(X,X) = 0$ $\forall i \geq 1$, ie X has no self-extensions. There are, however, many non-projective modules without self-extensions.

~~7.8~~ Conjecture (Auslander and Reiten): If $\mathrm{Ext}^i_A(X, X \oplus A) = 0$ $\forall i \geq 1$, then X is projective. (Open problem.)

(This is often refered to as "the" Auslander-Reiten conjecture, but at least one other conjecture by Auslander and Reiten also is called "the" Auslander-Reiten conjecture.)

A variant of global dimension considers modules of finite projective dimension only:

7.8 Definition: The finitistic dimension of A is

$\mathrm{findim}(A) := \sup\{\mathrm{pdim}(M) : M \in A\text{-mod and } \mathrm{pdim}(M) < \infty\}$.

Of course, $\mathrm{findim}(A) \leq \mathrm{gldim}(A)$ and in case A has finite global dimension, $\mathrm{findim}(A) = \mathrm{gldim}(A)$.

Finitistic dimension conjecture (Bass, 1960): $\forall A$: $\mathrm{findim}(A) < \infty$ (Open problem.)

(There also is $\mathrm{Findim}(A)$, where $M \in A$-Mod, not necessarily finitely generated, and a conjecture $\mathrm{Findim}(A) < \infty$ $\forall A$. There are known examples of algebras A with $\mathrm{findim}(A) < \mathrm{Findim}(A) < \infty$, and $\mathrm{Findim}(A) - \mathrm{findim}(A)$ as big as one wants.)

According to 7.4(d), $\mathrm{findim}(A) = 0$ for A self-injective. These algebras, with projective = injective are both frequent and kind of special. When the injective resolution of a regular module $_A A$ has length zero, then $_A A$ is injective and the algebra A is self-injective. Can one detect ~~from~~ self-injective from a longer injective resolution? Or from properties of the terms in an injective resolution?

Nakayama Conjecture (Nakayama, 1958): Suppose $_A A$ has an injective resolution where all terms are projective (as well as injective). Then A is self-injective. (Open problem.)

So, a minimal projective resolution with all terms projective and injective must have all but the first term zero.

There is ~~already~~ also a conjecture about injective resolutions of regular modules for A any algebra:

<u>Generalised Nakayama conjecture</u> (Auslander and Reiten, 1975): Let A be any algebra and I an indecomposable injective A-module. Then I is isomorphic to a direct summand of a term in the minimal injective resolution of ${}_AA$. In other words: all indecomposable injective A-modules occur somewhere in the minimal injective resolution of ${}_AA$. (Open problem.)

<u>7.9 Exercise:</u> (a) Show that the generalised Nakayama conjecture implies the Nakayama conjecture.
(b) Show that the finitistic dimension conjecture implies the Nakayama conjecture. (Hint: consider the syzygies $\Omega^{-n}({}_AA)$.)
(c) Show that the generalised Nakayama conjecture is valid for all A $\iff$ $\forall$ algebras B $\forall$ simple B-module S $\exists i \geq 0$: $\mathrm{Ext}^i_B(D(B_B), S) \neq 0$.

There is a system of such conjectures called "the homological conjectures" in representation theory of finite dimensional algebras." These are considered to be among the hardest and most important problems in the area. The finitistic dimension conjecture is at the top of this system, implying all others. The Nakayama conjecture is a bit lower down, but also very popular. The generalised Nakayama conjecture is equivalent to "the" Auslander-Reiten conjecture on page 7.5 (a result of Auslander and Reiten in 1975 shows the equivalence). (Here, like in many cases in this system, equivalence of two conjectures or implications between these means: Conjecture 1 valid $\forall A$ $\Rightarrow$ conjecture 2 valid $\forall A$, but not necessarily: conj 1 valid for A $\Rightarrow$ conj 2 valid for A.)
All conjectures have been verified for some classes of algebras, and of course the higher a conjecture is in the system, the smaller the evidence is for it to be true.