Before we continue, let us look back at our goals and at the previous parts of this chapter.

Our goals are: • define $Ext_A^i(X,Y)$ (where $X, Y \in A$-mod) for all i, so that Ext^1 becomes a special case

• Find long exact sequences of the form

$0 \to Hom \to Hom \to Hom \to Ext^1 \to Ext^1 \to Ext^1 \to Ext^2 \to Ext^2 \to Ext^2 \to \ldots$

such that the sequences in Theorems 3.9 and 3.10 are parts of such long exact sequences.

So we have to generalise from $i=1$ to all $i \geq 1$.

In the first part of this chapter we have seen situations with information in infinitely many degrees and we have seen infinitely long exact sequences: A (co)chain complex has (co)homology in infinitely many degrees. And by Theorem 6.5, a short exact sequence of (co)chain complexes gives us a long exact (co)homology sequence, which involves a connecting homomorphism (as in 3.3, 3.9 and 3.10).

To each module we can associate projective resolutions (which are chain complexes) and injective resolutions (which are cochain complexes). By the Horseshoe Lemma 6.9, these can be chosen to be compatible with given short exact sequences. The Comparison Theorem 6.13 is a tool to compare different resolutions and eventually to prove independence of certain data of the choice of resolutions.

Resolutions are complexes, but they are exact and hence their (co)homology vanishes in each degree. We better look at more interesting complexes.

Recall the basic idea in defining Ext^1: Hom is in ~~generality~~ general not exact, but Ext^1 "repairs" this non-exactness. This idea has been made precise in Theorem 3.3 and also in 3.9 and 3.10.

Idea: Instead of applying Hom to a short exact sequence, apply it to a long one, such as a projective or injective resolution.

We do an experiment to see what may happen:

Let A be a ring, X and Y A-modules. Choose a projective resolution P_* of X.

$$P_*: \dashrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0=\pi} X \to 0 \to \text{(notation as a chain complex)}$$

Drop X and get the chain complex $\mathring{P}_*: \dashrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0 \to 0 \to -$

Apply the contravariant Hom-functor $Hom_A(-,Y)$ and get a cochain complex

$$C^*: \dashrightarrow 0 \to Hom_A(P_0,Y) \xrightarrow{d_1^*} Hom_A(P_1,Y) \xrightarrow{d_2^*} Hom_A(P_2,Y) \xrightarrow{d_3^*} -$$

This complex is, in general, not exact. What is its cohomology?

Determine $H^0(C^*)$: Let $K := Ker(d_0)$, i.e. $0 \to K \to P \to X \to 0$ is exact.

$\Rightarrow 0 \to Hom_A(X,Y) \to Hom_A(P_0,Y) \to Hom_A(K,Y)$ is exact

and $d_1^* \searrow$ $\circlearrowleft$ $\int$ inj (since d_1 factors:
$$\begin{tikzcd} P_1 \arrow[r, "d_1"] \arrow[dr] & P_0 \\ & K \arrow[u, hook] \end{tikzcd}$$
)

$Hom_A(P_1,Y)$

$\Rightarrow Ker(d_1^*) = Hom_A(X,Y)$

$Image(0 \to Hom_A(P_0,Y)) = 0$ $\Rightarrow$ $\boxed{H^0(C^*) = Hom_A(X,Y)}$

Determine $H^1(C^*) = Ker(d_2^*)/Im(d_1^*)$

$d_2^*: Hom_A(P_1,Y) \to Hom_A(P_2,Y)$

$\alpha \longmapsto \alpha \circ d_2: P_2 \xrightarrow{d_2} P_1 \xrightarrow{\alpha} Y$

$P_1 \to Y$

$\alpha \in Ker(d_2^*)$ means $\alpha \circ d_2 = 0$, so α vanishes on $Im(d_2) = Ker(d_1)$

$\Rightarrow \alpha$ factors:
$$\begin{tikzcd} P_1 \arrow[r, "\alpha"] \arrow[dr] & Y \\ & P_1/Im(d_2) \arrow[u, "\bar{\alpha}"'] \end{tikzcd}$$

But $P_1/Im(d_2) = P_1/Ker(d_1) = Im(d_1) = K$

$\rightsquigarrow \bar{\alpha}: K \to Y$ and because of the factorisation of α, $Ker(d_2^*) = Hom_A(K,Y) \supset Im(d_1^*)$

Consider $0 \to Hom_A(X,Y) \xrightarrow{d_0^*} Hom_A(P_1,Y) \xrightarrow{d_1^*} Hom_A(K,Y) \to Coker\, d_1^* \to 0$

This sequence is exact, and $H^1(C^*)$ is isomorphic to $Coker\, d_1^*$.

$$\underset{Ker(d_2^*)/Im\, d_1^*}{\overset{\shortparallel}{H^1(C^*)}}$$

Compare with Theorem 3.3 and get: $\boxed{H^1(C^*) = Ext_A^1(X,Y)}$

6.14 Definition: Let A be a ring, X and Y A-modules and
$P_*: \ldots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to X \to 0$ a projective resolution of X. Let
$C^*: \ldots \to 0 \to \mathrm{Hom}_A(P_0,Y) \xrightarrow{d_1^*} \mathrm{Hom}_A(P_1,Y) \xrightarrow{d_2^*} \mathrm{Hom}_A(P_2,Y) \to \ldots$ (a cochain complex).
Let $i \in \mathbb{N}_0$. Then $H^i(C^*) =: \mathrm{Ext}_A^i(X,Y)$ is called the <u>i-th extension group</u>.

So, $\mathrm{Hom} = \mathrm{Ext}^0$ and $\mathrm{Ext}^1 = \mathrm{Ext}^1$, and Ext^i is an abelian group (or more).
Alternatively, we could have started with an injective resolution of Y:
$0 \to Y \to I^0 \to I^1 \to I^2 \to \ldots$ (a cochain complex I^*), dropped Y to get
$\dot{I}^*: 0 \to I^0 \to I^1 \to I^2 \to \ldots$ and then apply the covariant $\mathrm{Hom}_A(X,-)$ to get
$D^*: 0 \to \mathrm{Hom}_A(X,I^0) \to \mathrm{Hom}_A(X,I^1) \to \mathrm{Hom}_A(X,I^2) \to \ldots$ (cochain complex)
Its cohomology $H^i(D^*)$ is isomorphic to $\mathrm{Ext}_A^i(X,Y)$, which thus can be computed in two different ways.
We are not going to check this, or independence of the choice of resolution.
Instead we continue to develop abstract theory and doing so will solve the problems: are $\mathrm{Ext}_A^i(X,-)$ and $\mathrm{Ext}_A^i(-,Y)$ functors?

do the long exact cohomology sequences on short exact sequences really continue the sequences in 3.9 and 3.10?

Then we have reached our goals. There is, however, a question remaining:
Is our solution, i.e. our definition of Ext^i for all i, unique in some sense, or even best possible?
This is related to another question: What properties does the family of functors $\mathrm{Ext}_A^i(X,-)$ (for $i \geq 0$) or the family of functors $\mathrm{Ext}_A^i(-,Y)$ have?

The machinery to be developed works well when starting with Hom functors, which as we know are left exact. It always works well with right exact functors, which we still have to find. There is a family of such functors that are the counterparts of Hom, and later on we will find these, too.

First we make precise the properties families of functors like Ext^n are supposed to have:

6.15 Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $T_n : \mathcal{A} \to \mathcal{B}$, for $n \in \mathbb{N}_0$, a family of covariant additive functors. Suppose that for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ and each n, a morphism $\delta_n : T_n(C) \to T_{n-1}(A)$ is given.

All T_n and δ_n together are forming a (covariant) homological δ-functor $:\Leftrightarrow$

(1) For each ses $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ there is a long exact sequence in $\mathcal{B}$:

$$\dots \to T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \to \dots \to T_0(C) \to 0 =: T_{-1}(A)$$

(2) For each morphism of short exact sequences in $\mathcal{A}$

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & B' & \to & C' & \to & 0 \\ & & f\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow g & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

there are commutative diagrams in $\mathcal{B}$, for all n.

$$\begin{array}{ccc} T_n(C') & \xrightarrow{\delta_n} & T_{n-1}(A') \\ T_n(f)\downarrow & \circlearrowleft & \downarrow T(g) \\ T_n(C) & \longrightarrow & T_{n-1}(A) \end{array}$$

A (covariant) cohomological δ-functor consists of a collection of covariant additive functors $T^n : \mathcal{A} \to \mathcal{B}$ for $n \geq 0$ (and $T^n = 0$ for $n < 0$) together with morphisms $\delta^n : T^n(C) \to T^{n+1}(A)$ for each ses $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, such that

(1) for each ses $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ there is a long exact sequence in $\mathcal{B}$

$$\dots \to T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \to \dots$$

(2) for each morphism of ses in $\mathcal{A}$:

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & B' & \to & C' & \to & 0 \\ & & f\downarrow & & \downarrow & & \downarrow g & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

there are commutative diagrams in $\mathcal{B}$, for all n

$$\begin{array}{ccc} T^n(C') & \to & T^{n+1}(A') \\ \downarrow & \circlearrowleft & \downarrow T^{n+1}(f) \\ T^n(C) & \to & T^{n+1}(A) \end{array}$$

(Contravariant (co)homological δ-functors are defined analogously.)

δ-functors are not functors, but collections of functors and further data. We write T for the whole collection.

Since the T_n and the T^n are functors, all diagrams except those in (2) are commutative automatically.
By (1), T_0 is right exact and T^0 is left exact; we will choose a Hom-functor for T^0.

Examples of (co)homological δ-functors:

- all $T_n = 0$ or all $T^n = 0$, and all δ_n or δ^n also zero
- $\mathcal{A} = Ch_{\geq 0}(\mathcal{B})$ the category of chain complexes concentrated in non-negative degrees. H_* (taking homology) yields a homological δ-functor $H_*: \mathcal{A} \to \mathcal{B}$.
  Cohomology $H^*: \mathcal{A} \to \mathcal{B}$ yields a cohomological δ-functor.
- A different example of a homological δ-functor. Let $\mathcal{A} = Ab$, the category of abelian groups. Fix $k \in \mathbb{Z}$ and let $\varphi_k : A \to A$, $a \mapsto ka$ for each abelian group A.
  Let $T_0(A) := A/kA$ and
  $T_1(A) := \{a \in A : ka = 0\}$. check that T_0 and T_1 are functors
  Note that $T_1(A) = $ kernel ~~of~~ of φ_k and $T_0(A) = $ cokernel of φ_k.
  Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Then there is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & T_1(A) \arrow[r] \arrow[d, "\text{incl}"] & T_1(B) \arrow[r] \arrow[d, "\text{incl}"] & T_1(C) \arrow[d, "\text{incl}"] \arrow[dddll, dashed, bend left, "\delta_1"] \\
0 \arrow[r] & A \arrow[r] \arrow[d, "k"] & B \arrow[r] \arrow[d, "k"] & C \arrow[d, "k"] \\
0 \arrow[r] & A \arrow[r] \arrow[d, "\text{proj}"] & B \arrow[r] \arrow[d, "\text{proj}"] & C \arrow[d, "\text{proj}"] \\
 & T_0(A) \arrow[r] & T_0(B) \arrow[r] & T_0(C)
\end{tikzcd}$$

where δ_1 comes from the Snake Lemma.

$(T_0, T_1, 0, 0, \dots)$ forms a homological δ-functor. verify

There are many (co)homological δ-functors. To get uniqueness, one has to require further properties.

6.16 Definition: A homological δ-functor $T: \mathcal{A} \to \mathcal{B}$ is universel $:\Leftrightarrow$ $\forall$ homological δ-functors $S: \mathcal{A} \to \mathcal{B}$ and for all natural transformations $f_0: S_0 \to T_0$ $\exists!$ morphism $f: S \to T$ of δ-functors extending f_0.
A cohomological δ-functor $T: \mathcal{A} \to \mathcal{B}$ is universal $:\Leftrightarrow$ $\forall$ cohomological δ-functors $S: \mathcal{A} \to \mathcal{B}$ and for all natural transformations $f^0: T^0 \to S^0$ $\exists! f: T \to S$, morphism of δ-functors, extending f^0.
(Mind the difference: $f_0: S_0 \to T_0$ while $f^0: T^0 \to S^0$.)

A morphism of δ-functors is a sequence of natural transformations like $f_n: S_n \to T_n$ which are compatible with each other with respect to the conditions (1) and (2).

As usual, the uniqueness required by the universal property implies uniqueness of T once T_0 is fixed. If S also satisfies the condition wrt $S_0 = T_0$, then $id: S_0 \xrightarrow{f_0} T_0$ can be lifted uniquely to $f: S \to T$ which by uniqueness must be a sequence of natural isomorphisms.
So, when we prove that Ext^n form a universal δ-functor with degree 0 form Hom, we have shown that Ext^n is the best way to extend Hom to a series of functors. "Best" is defined by the properties in definitions 6.15 and 6.16.
An easy example of a universal δ-functor: Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor. Choose $T_0 = F$ and $T_n = 0$ for $n > 0$, or $T^0 = F$ and $T^n = 0$ for $n > 0$.
Since there is exactly one map $S_n \to 0$ or $0 \to S^n$, this is a universal δ-functor. Conditions (1) and (2) are satisfied as F is exact (but otherwise they wouldn't). verify the details
This fits to our idea to use Ext^n to correct the lack of exactness of Hom.
For P projective, I injective, the functors $Hom(P,-)$ and $Hom(-,I)$ are exact and there is nothing to be corrected.

Since T_0 has to be right exact and T^0 has to be left exact, these are the two situations we want to work out now, for abelian categories $\mathcal{A}, \mathcal{B}$. Let $F: \mathcal{A} \to \mathcal{B}$ be given and either left exact or right exact. In the first case we assume that $\mathcal{A}$ has enough injective objects, in the second case we assume existence of enough projective objects. The assumptions allow us to work with injective or projective resolutions. For $\mathcal{A} = R\text{-Mod}$ these assumptions are no restrictions.
The following constructions fit to our definition of Ext^i.

<u>Case F left exact:</u> For an object A in $\mathcal{A}$, let I^* be an injective resolution of A, written as a cochain complex $\ldots \to 0 \to 0 \to I^0 \to I^1 \to I^2 \to \ldots$
Apply F to get a complex $\ldots \to F(0) \to F(0) \to F(I^0) \to F(I^1) \to F(I^2) \to \ldots$ (if F is contravariant: $\ldots \leftarrow F(0) \leftarrow F(I^0) \leftarrow \ldots$), which may have non-vanishing cohomology. $F(I^*)$ is our notation for this complex.
Set $(R^iF)(A) := H^i(F(I^*))$ and call R^iF the <u>i-th right derived functor</u> of F. Since F is left exact, $(R^0F)(A) = H^0(F(I^*)) \underset{\text{why?}}{=} F(A)$
$\Rightarrow R^0F = F$.

<u>Case F right exact:</u> For an object A in $\mathcal{A}$, let P_* be a projective resolution of A, written as a chain complex $\ldots \to P_2 \to P_1 \to P_0 \to 0 \to 0 \to \ldots$
Apply F to get a complex $\ldots \to F(P_2) \to F(P_1) \to F(P_0) \to \underset{0}{\underset{\shortparallel}{F(0)}} \to F(0) \to \ldots$ (covariant F), called $F(P_*)$ (and similarly in the contravariant case).
Set $(L_iF)(A) := H_i(F(P_*))$ and call L_iF the <u>i-th left derived functor</u> of F. Since F is right exact, $(L_0F)(A) = H_0(F(P_*)) = F(A)$.

Since F, H_0 and H^0 are functors, R^0F and L_0F coincides with F not only on objects, and – since H_i and H^i are functors, too – R^iF and L_iF are functors.
Checking that these are δ-functors requires quite some work. But the definitions are explicit and examples can be computed explicitly.

(These constructions can be carried out for any functor F, not necessarily left or right exact. But we need left or right exact, respectively, for the following result.)

6.17 Theorem: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F: \mathcal{A} \to \mathcal{B}$ a covariant functor, or a contravariant functor.
(a) Suppose F is left exact and $\mathcal{A}$ has enough injectives. Then the left derived functors $L_i F$ $(i \geq 0)$ form a universal homological δ-functor.
(b) Suppose F is left exact and $\mathcal{A}$ has enough projectives. Then the right derived functors $R^i F$ $(i \geq 0)$ form a universal cohomological δ-functor.

The proof of 6.17 is quite long and technical. It has been outsourced to an appendix to this chapter. It's worth reading it, but it is too technical to be an exam topic.

Let's compute Ext^n in an example: k a field, $A = k[x]/\langle x^2 \rangle$ k-algebra, $k = A/\langle x \rangle$ simple module. A is indecomposable projective and indecomposable injective. There is a short exact sequence $0 \to k \to A \to k \to 0$.
We will compute $\mathrm{Ext}^n_A(k,k)$ twice – once using a projective resolution of k and once using an injective resolution. And then we compute $\mathrm{Ext}^n_A(A,k)$ and $\mathrm{Ext}^n_A(k,A)$.

$$\text{Coker} = k$$

Injective resolution: $0 \to k \xrightarrow[\bar{1} \mapsto x]{} A \xrightarrow[1 \mapsto x]{} A \xrightarrow[1 \mapsto x]{} A \xrightarrow[1 \mapsto x]{} A \to \ldots$ (this happens to be periodic

$\rightsquigarrow \overset{\circ}{I}{}^*: \ldots \to 0 \to 0 \to A \xrightarrow{1 \mapsto x} A \xrightarrow{1 \mapsto x} A \xrightarrow{1 \mapsto x} A \to \ldots$

Apply $\mathrm{Hom}_A(k,-)$ $(=F)$ and get $D^*: \to 0 \to \mathrm{Hom}_A(k,A) \to \mathrm{Hom}_A(k,A) \to \mathrm{Hom}_A(k,A) \to$
The map $\mathrm{Hom}_A(k,A) \longrightarrow \mathrm{Hom}_A(k,A)$ always is the same: composing with $A \xrightarrow[1 \mapsto x]{\varphi} A$

Let $\psi: k \to A$ be an A-module homomorphism.
$x \cdot \bar{1} = 0$ in k $\Rightarrow x\,\psi(\bar{1}) = 0$ in $A \Rightarrow \overset{\psi}{\cancel{\varphi}}(\bar{1}) = \lambda x \in A$ for some $\lambda \in k$.
Then $\varphi \circ \psi: \bar{1} \overset{\psi}{\longmapsto} \underset{\lambda x \cdot 1}{\lambda x} \longmapsto \lambda x^2 \cdot 1 = 0 \Rightarrow \mathrm{Hom}_A(k,A) \xrightarrow{0} \mathrm{Hom}_A(k,A)$ everywhere

$\Rightarrow F(I^*)/Ts: \; 0 \to \underset{\substack{\cong \\ k}}{Hom_A(k,A)} \xrightarrow{0} \underset{\substack{\cong \\ k}}{Hom_A(k,A)} \xrightarrow{0} \underset{\substack{\cong \\ k}}{Hom_A(k,A)} \xrightarrow{0} \dots$

Its cohomology ~~Hks~~ is k in each degree.

$\Rightarrow \; Ext^i_A(k,k) = k \;\forall i \geq 0$. This confirms $\underset{\substack{\cong \\ Hom_A(k,k)}}{Ext^0_A(k,k)} = k = Ext^1_A(k,k)$

Projective resolution: $\dots \to A \xrightarrow{1 \mapsto x} A \xrightarrow{1 \mapsto x} A \xrightarrow{1 \mapsto x} A \xrightarrow{1 \mapsto \bar{1}} k \to 0 \to 0 \to \dots$

$Ker = k \cong \langle x \rangle$

$\rightsquigarrow P_*: \; \dots \to A \to A \to A \to A \to A \to 0 \to 0 \to \dots$

Apply $Hom_A(-,k)$ (contravariant) and get

$C^*: \to 0 \to 0 \to \underset{\substack{\cong \\ k}}{Hom_A(A,k)} \to \underset{\substack{\cong \\ k}}{Hom_A(A,k)} \to \underset{\substack{\cong \\ k}}{Hom_A(A,k)} \to \dots$

The maps $Hom_A(A,k) \to Hom_A(A,k)$ are precomposition with $\psi: A \to A$, $1 \mapsto x$

So, for $\varphi: A \to k$, we get $\varphi \mapsto \varphi \circ \psi: A \xrightarrow{1 \mapsto x} A \xrightarrow{1 \mapsto \bar{1}} k$, i.e. $\varphi \circ \psi = 0$

$\rightsquigarrow \; \dots \to 0 \to k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} \dots$ Cohomology is k in each degree, as it should be, since this again computes $Ext^i_A(k,k)$, as above.

Now we apply property (1) in 6.15 to the short exact sequence $0 \to k \to A \to k \to 0$ with respect to the functors $Hom_A(-,k)$ and $Hom_A(k,-)$:

$Hom_A(k,-) \rightsquigarrow \; 0 \to \underset{\substack{\cong \\ k}}{Hom_A(k,k)} \to \underset{\substack{\cong \\ k}}{Hom_A(k,A)} \to \underset{\substack{\cong \\ k}}{Hom_A(k,k)} \to \underset{\substack{\cong \\ k}}{Ext^1_A(k,k)} \to$

$\to \underset{\substack{\cong \\ 0}}{Ext^1_A(k,A)} \to \underset{\substack{\cong \\ k}}{Ext^1_A(k,k)} \to \underset{\substack{\cong \\ k}}{Ext^2_A(k,k)} \to Ext^2_A(k,A) \to$

$\to \underset{\substack{\cong \\ k}}{Ext^2_A(k,k)} \to \underset{\substack{\cong \\ k}}{Ext^3_A(k,k)} \to Ext^3_A(k,A) \to \underset{\substack{\cong \\ k}}{Ext^3_A(k,k)} \to \dots$

We know the Hom's and all $Ext^i_A(k,k)$. Also $Ext^1_A(k,A) = 0$ since A is injective.

By exactness, the sequence must start in the following way

$0 \to k \xrightarrow{\simeq} k \xrightarrow{0} k \xrightarrow{\simeq} k \to 0 \to k \simeq k \xrightarrow{0}$

How to continue and identify the remaining vectorspaces and linear maps?

A is injective ⇒ its injective resolution is $0 \to A \xrightarrow{id} \underbrace{A \to 0 \to 0 \to \dots}_{\text{inj resolution } I}$

and we can use this to compute $Ext^i(k,A)$:

Apply $Hom_A(k,-)$ to this injective resolution and get

$$0 \to \underbrace{Hom_A(k,A)}_{\text{degree } 0} \to \underbrace{Hom_A(k,0) \to Hom_A(k,0) \to \dots}_{\text{degrees } i \geq 1}$$

$\Rightarrow Ext^i_A(k,A) = 0 \ \forall i \geq 1$

Plugging these zeros into the long exact sequence we get

$$0 \to \underbrace{k \xrightarrow{\sim} k \xrightarrow{0} k}_{Hom} \xrightarrow{\sim} \underbrace{k \to 0 \to k}_{Ext^1} \xrightarrow{\sim} \underbrace{k \xrightarrow{0} 0 \to k}_{Ext^2} \xrightarrow{\sim} \underbrace{k \to 0 \to k}_{Ext^3} \xrightarrow{\sim} k \to \dots$$

which continues like that forever.

Applying $Hom_A(-,k)$ instead we get:

$$0 \to \underset{\cong k}{Hom_A(k,k)} \to \underset{\cong k}{Hom_A(A,k)} \to \underset{\cong k}{Hom_A(k,k)} \to \underset{\cong k}{Ext^1_A(k,k)} \to \underset{\cong 0}{Ext^1_A(A,k)} \to$$

$$\to \underset{\cong k}{Ext^1_A(k,k)} \to \underset{\cong k}{Ext^2_A(k,k)} \to \underset{\cong 0}{Ext^2_A(A,k)} \to \underset{\cong k}{Ext^2_A(k,k)} \to \underset{\cong k}{Ext^3_A(k,k)} \to \dots$$

where $Ext^i_A(A,k) = 0$ for $i \geq 1$ since it can be computed using the projective resolution $\to 0 \to 0 \to A \xrightarrow{id} A \to 0$ of A.

So, the long exact sequence is

$$0 \to \underbrace{k \xrightarrow{\sim} k \xrightarrow{0} k}_{Hom} \simeq \underbrace{k \to 0 \to k}_{Ext^1} \xrightarrow{\sim} \underbrace{k \to 0 \to k}_{Ext^2} \xrightarrow{\sim} \underbrace{k \to 0 \to k}_{Ext^3} \xrightarrow{\sim} \underbrace{k \to 0}_{Ext^4} \to \dots$$

In both cases, we could quickly identify the morphisms in the long exact cohomology sequence (as 0 or as isomorphism) just from exactness of the sequence and the k-dimension of the Ext spaces.

In the examples we have shown already:

6.18 Proposition: Let A be a ring and P and I A-modules.
(a) When P is projective, then $\mathrm{Ext}_A^i(P,M)=0$ $\forall i\geq 1$ $\forall$ A-modules M.
(b) When I is injective, then $\mathrm{Ext}_A^i(M,I)=0$ $\forall i\geq 1$ $\forall$ A-modules M.
(The converse statements are true as well: this characterises projective and injective modules, respectively.)

In the examples, the vanishing of $\mathrm{Ext}_A^i(P,-)$ or $\mathrm{Ext}_A^i(-,I)$ forced the neighbouring extension groups to be isomorphic. This is a general fact:
Let $0\to\Omega_1 X\to P\to X\to 0$ be exact and apply $\mathrm{Hom}_A(-,Y)$:

$$0\to \mathrm{Hom}_A(X,Y)\to\mathrm{Hom}_A(P,Y)\to\mathrm{Hom}_A(\Omega_1X,Y)\to\mathrm{Ext}_A^1(X,Y)\to\underbrace{\mathrm{Ext}_A^1(P,Y)}_{=0}\to$$

$$\to\mathrm{Ext}_A^1(\Omega_1X,Y)\to\mathrm{Ext}_A^2(X,Y)\to\underbrace{\mathrm{Ext}_A^2(P,Y)}_{=0}\to\mathrm{Ext}_A^2(\Omega_1X,Y)\to\mathrm{Ext}_A^3(X,Y)\to\dots$$

Then for $i\geq 1$: $\mathrm{Ext}_A^i(\Omega_1X,Y)\simeq\mathrm{Ext}_A^{i+1}(X,Y)$ because of the neighbouring 0's in the long exact sequence. Similarly in the case of I.

6.19 Proposition: Let $0\to\Omega_1X\to P\to X\to 0$ exact with P projective and $0\to X\to I\to\Omega^1X\to 0$ exact with I injective and M any module.
Then $\forall i\geq 1$: $\mathrm{Ext}_A^i(\Omega_1X,M)\simeq\mathrm{Ext}_A^{i+1}(X,M)$

and $\mathrm{Ext}_A^i(M,\Omega^1X)\simeq\mathrm{Ext}_A^{i+1}(M,X)$

These statements often are called <u>dimension shift</u>. We will understand this name better when we relate Ext with projective or injective dimensions, respectively.