For us, important examples of complexes are modules – seen as complexes concentrated in degree zero – and their projective or injective resolutions (which we know to be quasi-isomorphic to the module being resolved). Projective or injective resolutions are ~~far~~ far from being unique, and there are some practical question we need to address:

- Is there a "smallest" projective module P mapping onto a given M? Or a smallest injective module I containing M?
- Are two projective (or injective) resolutions of M always related by (co)chain maps and thus can be compared?
- Given a short exact sequence $0 \to X \to Y \to Z \to 0$, suppose we know projective (or injective) resolutions of both X and Z. Can we construct a resolution for Y from the given resolutions? Does the idea in the proof of Theorem 3.10 work here again?

We start with the first question. In general, the answer is negative. One fails already defining "smallest", when the ring fails for instance unique decomposition (Krull–Remak–Schmidt) – ${}_RR \simeq {}_RR^n$ $\forall n \in \mathbb{N}$ may happen. For finite dimensional algebras, the situation is much better.

6.6 Definition: Let A be a finite dimensional K-algebra and M a finite dimensional A-module.

A projective cover of M is an epimorphism $P \xrightarrow{f} M$ with P projective such that each endomorphism $\alpha \in \mathrm{End}_A(P)$ satisfying $f \circ \alpha = f$ is an isomorphism.

An injective hull (or injective envelope) of M is a monomorphism $M \xrightarrow{g} I$ with I injective such that endomorphism $\beta \in \mathrm{End}_A(I)$ satisfying $\beta \circ g = g$ is an isomorphism.

The desired minimality becomes clearer in the following characterisation of covers / hulls:

6.7 Proposition: For A and M finite dimensional, the following assertions are equivalent for $f: P \to M$ (projective) and, respectively, for $g: M \to I$ (injective):

(a) $f: P \to M$ is a projective cover.

(b) For each epimorphism $\tilde{f}: Q \to M$ with projective Q $\exists$ epimorphism $\pi: Q \to P$ such that $f \circ \pi = \tilde{f}$. In a diagram

$$\begin{tikzcd} P \arrow[r, two heads, "f"] & M \\ & Q \arrow[ul, dashed, two heads, "\exists \pi"] \arrow[u, two heads, "\forall \tilde{f}"'] \end{tikzcd}$$

(c) For each epimorphism $\tilde{f}: Q \to M$ with projective Q, there is a direct summand of Q isomorphic to P.

(d) For each epimorphism $\tilde{f}: Q \to M$ with Q projective there is an inequality of k-dimensions $\dim_k P \leq \dim_k Q$.

(a') $g: M \to I$ is an injective hull

(b') For each monomorphism $\tilde{g}: M \hookrightarrow J$ with injective J $\exists$ monomorphism $\iota: I \to J$ such that $\tilde{g} = \iota \circ g$

(c') For each monomorphism $\tilde{g}: M \hookrightarrow J$ with injective J, there is a direct summand of J isomorphic to J.

(d') For each monomorphism $\tilde{g}: M \hookrightarrow J$ with J injective, there is an inequality of k-dimensions $\dim_k I \leq \dim_k J$.

(So, covers and hulls actually are minimal in several ways.)

Proof: The four statements on covers are dual to the four statements on hulls, and ~~$D(A_A)$~~ direct summands of finite direct sums of $D(A_A)$ are the injectives in A-mod. $\Rightarrow$ It is sufficient to prove the equivalences for injective hulls. Dualising then yields the proof for projective covers.

First, let us relate endomorphisms $\beta \in \mathrm{End}_A(I)$ with $\tilde{g}$ as in the other conditions:

Given $\begin{tikzcd} M \arrow[r, hook, "g"] \arrow[dr, "\tilde{g}"'] & I \\ & J \end{tikzcd}$ with I and J injective modules: Since I is injective, $\tilde{g}$ factors ~~thau~~ through I: $\begin{tikzcd} M \arrow[r, hook, "g"] \arrow[dr, "\tilde{g}"'] & I \arrow[d, "\exists \varphi"] \\ & J \end{tikzcd}$ $\exists \varphi: I \to J$ such that $\tilde{g} = \varphi \circ g$

Since J is injective, g factors through J: $\begin{tikzcd} M \arrow[r, hook, "g"] \arrow[dr, "\tilde{g}"'] & I \\ & J \arrow[u, "\exists \psi"'] \end{tikzcd}$ $\exists \psi: J \to I$ such that $g = \psi \circ \tilde{g}$

Note that φ or ψ need not be injective (= mono), but (b') requires there to exist a mono $\varphi = \iota: I \to J$.
The composition $\psi \circ \varphi$ is an endomorphism of I and it satisfies $(\psi\circ\varphi)\circ g = \psi\circ(\varphi\circ g) = \psi \circ \tilde{g} = g$ as required in (a'), the definition of hull.
This tells us how to prove (a') $\Rightarrow$ (b'): By (a'), $\psi\circ\varphi$ is an isomorphism $\Rightarrow$ φ is injective = mono $\Rightarrow$ (b').
(b') $\Rightarrow$ (c'): Since $\iota: I \hookrightarrow J$ is mono, and I is injective, ι splits and I is, through ι, isomorphic to a direct summand of J.
(c') $\Rightarrow$ (d') is clear: the K-dimension of a summand is less or equal the K-dimension of J itself.
(d') $\Rightarrow$ (a'): Let $\beta \in \mathrm{End}_A(I)$ satisfy $\beta\circ g = g$. We have to show that β is an isomorphism. By Fitting's Lemma (see the recap on unique decompositions, page 2), $\exists\, n \in \mathbb{N}: I = \mathrm{Ker}(\beta^n) \oplus \mathrm{Im}(\beta^n)$. Direct summands of an injective module are injective, too.
By assumption, $\beta\circ g = g$ and g is injective. For $m \in M$, $g(m) = \beta(g(m)) = \beta^2(g(m)) = \ldots = \beta^n(g(m))$. $\Rightarrow \mathrm{Im}(g) \subset \mathrm{Im}(\beta^n) \Rightarrow g$ factors through $J := \mathrm{Im}(\beta^n)$:

$$\begin{tikzcd} M \arrow[r, hook, "g"] \arrow[dr, "\tilde{g}"'] & I = J \oplus \mathrm{Ker}(\beta^n) \\ & J \arrow[u] \end{tikzcd}$$

g mono $\Rightarrow \tilde{g}$ mono and $\dim_K J \le \dim_K I$

But $\dim_K I \le \dim_K J$ by (d') $\Rightarrow \dim_K I = \dim_K J \Rightarrow I = J$ and $I = \mathrm{Im}(\beta^n)$ $\Rightarrow \beta^n$ surjective $\Rightarrow \beta$ surjective $\underset{\dim I < \infty}{\Rightarrow}$ β isomorphism $\Rightarrow$ (a'). $\square$

This implies existence of projective covers and injective hulls, up to isomorphism:
<u>6.8 Corollary:</u> Let the algebra A and the A-module M be finite dimensional. Then M has a projective cover $P \twoheadrightarrow M$ and an injective hull $I \hookleftarrow M$. Both P and I are unique up to isomorphism.
Proof: Among all projectives mapping onto M, we choose P of minimal dimension. By 6.7 this gives a projective cover and by 6.7 (b) or (c), P is unique up to isomorphism. Analogously, I can be chosen of minimal K-dimension. $\square$

Next we fix a short exact sequence and build a projective or injective resolution of the middle term from those of the outer terms:

6.9 Horseshoe Lemma: Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of R-modules.

(a) Let $\ldots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} X \to 0$ and $\ldots \to Q_2 \xrightarrow{d_2'} Q_1 \xrightarrow{d_1'} Q_0 \xrightarrow{\pi'} Z \to 0$ be projective resolutions. Then there exist homomorphisms such that the following diagram is commutative with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d] & P_0 \arrow[r, "\pi"] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
\ldots \arrow[r, "\tilde{d}_3"] & P_2 \oplus Q_2 \arrow[r, "\tilde{d}_2"] \arrow[d] & P_1 \oplus Q_1 \arrow[r, "\tilde{d}_1"] \arrow[d] & P_0 \oplus Q_0 \arrow[r, "\tilde{\pi}"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
\ldots \arrow[r, "d_3'"] & Q_2 \arrow[r, "d_2'"] \arrow[d] & Q_1 \arrow[r, "d_1'"] \arrow[d] & Q_0 \arrow[r, "\pi'"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

(b) Let $0 \to X \xrightarrow{\iota} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \xrightarrow{d_3} \ldots$ and $0 \to Z \xrightarrow{\iota'} I_0 \xrightarrow{d_1'} I_1 \xrightarrow{d_2'} I_2 \xrightarrow{d_3'} \ldots$ be injective resolutions. Then there exist homomorphisms such that the following diagram is commutative with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d] & E_0 \arrow[r, "d_1"] \arrow[d] & E_1 \arrow[r, "d_2"] \arrow[d] & E_2 \arrow[r, "d_3"] \arrow[d] & {} \\
0 \arrow[r] & Y \arrow[r, "\tilde{\iota}"] \arrow[d] & E_0 \oplus I_0 \arrow[r, "\tilde{d}_1"] \arrow[d] & E_1 \oplus I_1 \arrow[r, "\tilde{d}_2"] \arrow[d] & E_2 \oplus I_2 \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & Z \arrow[r, "\iota'"] \arrow[d] & I_0 \arrow[r, "d_1'"] \arrow[d] & I_1 \arrow[r, "d_2'"] \arrow[d] & I_2 \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

The proof of 3.10 tells us how to start the proof of (a). We will work out the proof in the case of injective resolutions.

Proof (of (b)): The construction proceeds inductively, of course.

First step: Given

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, "f"] & E_0 \arrow[d] \\ & Y \arrow[d, "g"] & E_0 \oplus I_0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "\iota'"] \arrow[d] & I_0 \arrow[d] \\ & 0 & 0 \end{tikzcd}$$

Add $E_0 \oplus I_0$ into the 2^{nd} column and turn the 2^{nd} column into a split exact sequence, using the inclusion $E_0 \to E_0 \oplus I_0$ of the first summand and the projection $E_0 \oplus I_0 \to I_0$ onto the second summand.

Define $\tilde{\iota}: Y \to E_0 \oplus I_0$ as follows: $Y \xrightarrow{g} Z \xrightarrow{\iota'} I_0$ is the map $Y \to I_0$. Since E_0 is injective, $\iota: X \to E_0$ lifts to a map $Y \to E_0$ (choose any) such that

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "f"] & E_0 \\ Y \arrow[ur] \end{tikzcd}$$

commutes. Since the two squares involving $\tilde{\iota}$ commute, a variant of the short five lemma (outer ~~terms~~ maps injective $\Rightarrow$ middle map injective) shows that $\tilde{\iota}$ is injective. This can be checked directly as well: If $\iota'(g(y_0)) = 0$ then $\exists x_0: y_0 = f(x_0)$ and $\tilde{\iota}(y_0) = 0$ implies also $\iota(x_0) = 0$, hence $x_0 = 0$ and $y_0 = 0$.

This gives the diagram which we complete by the cokernels

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ \ker(\iota) = & 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d] & E_0 \arrow[r] \arrow[d] & \operatorname{cok}(\iota) = \operatorname{Im}(d_1) \arrow[d] \\ \ker(\tilde{\iota}) = & 0 \arrow[r] & Y \arrow[r, "\tilde{\iota}"] \arrow[d] & E_0 \oplus I_0 \arrow[r] \arrow[d] & \operatorname{cok}(\tilde{\iota}) (= \operatorname{Im}(\tilde{d}_1) \text{ if } \tilde{d}_1 \text{ exists}) \arrow[d] \\ \ker(\iota') = & 0 \arrow[r] & Z \arrow[r, "\iota'"] \arrow[d] & I_0 \arrow[r] \arrow[d] & \operatorname{cok}(\iota') = \operatorname{Im}(d_1') \arrow[d] \\ & & 0 & 0 & 0 \end{tikzcd}$$

The right hand column is exact (this is part of the Snake Lemma) and by exactness of the given resolutions, $\operatorname{cok}(\iota) = \operatorname{Im}(d_1)$ and $\operatorname{cok}(\iota') = \operatorname{Im}(d_1')$.

The given maps d_1 and d_1' factor through their images

$$\begin{tikzcd} E_0 \arrow[rr, "d_1"] \arrow[dr] & & E_1 \\ & \operatorname{cok}(\iota) = \operatorname{Im}(d_1) \arrow[ur] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I_0 \arrow[rr, "d_1'"] \arrow[dr] & & I_1 \\ & \operatorname{cok}(\iota') = \operatorname{Im}(d_1') \arrow[ur] \end{tikzcd}$$

which gives the situation

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{Im}(d_1) = \operatorname{cok}(\iota) \arrow[r] \arrow[d] & E_1 \arrow[d] \\
0 \arrow[r] & \operatorname{cok}(\tilde{\iota}) \arrow[r] \arrow[d] & E_1 \oplus I_1 \arrow[d] \\
0 \arrow[r] & \operatorname{Im}(d_1') = \operatorname{cok}(\iota') \arrow[r] \arrow[d] & I_1 \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

As in the first step, this can be completed to a commutative diagram with exact rows and columns. The proof continues by iterating the previous steps in this new situation.

$\square$

The resolutions of X provided by 6.9 are usually not the minimal ones that would be built by taking projective covers or injective hulls in every step, respectively. But it may be much easier to construct the resolutions in 6.9, and the diagrams in 6.9 may be useful, too. Generally we would lose flexibility by always insisting on minimal resolutions. Instead, we will now try to understand how different resolutions of the same module are related with each other.

In order to compare resolutions, it makes sense to define a new concept, homotopy of maps, and a new category. We abbreviate complexes C_* as C, D_* as D, –

<u>6.10 Definition:</u> Let C and D be chain complexes and $f, g, h: C \to D$ morphisms of complexes of R-modules.

(a) The morphism f is <u>null-homotopic</u> $:\Leftrightarrow$ $\exists$ sequence $\{s_n : C_n \to D_{n+1}\}_{n\in\mathbb{Z}}$ of homomorphisms of R-modules such that $\forall n$: $f_n = s_{n-1} \circ d_n + d_{n+1} \circ s_n$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] \arrow[dl, "s_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"'] \\
D_{n+1} \arrow[r, "d_{n+1}"'] & D_n \arrow[r, "d_n"'] & D_{n-1}
\end{tikzcd}$$

(abbreviate as $f = s\circ d + d \circ s$)

(b) g and h are <u>homotopic</u> $:\Leftrightarrow$ $g-h$ is null-homotopic, i.e. $\exists s: g-h = s\circ d + d\circ s$. s is then called a <u>homotopy</u> between g and h.

(c) f is called a <u>homotopy equivalence</u> $:\Leftrightarrow$ $\exists$ morphism $j: D \to C$ of complexes such that $f\circ j$ is homotopic to 1_D and $j \circ f$ is homotopic to 1_C.

The same definitions can be made for cochain complexes. One may talk of chain homotopic or cochain homotopic, etc, depending on the situation.

You may have seen homotopies in topology (=qualitative ~~Ge~~ geometry), in particular in algebraic topology, which is about functors from geometry to algebra. The wikipedia page on homotopy shows a homotopy relating a dougnot and a coffee mug.

An important property is that homotopic chain complexes have the same homology:

<u>6.11 Lemma:</u> Let C and D be chain complexes and $f,g: C \to D$ morphisms.
(a) If f is null-homotopic, then the homology $H_n(f) = 0$ $\forall n \in \mathbb{Z}$.
(b) If f and g are homotopic, then $H_n(f) = H_n(g)$ $\forall n \in \mathbb{Z}$.
(There are obviously analogous statements for cochain complexes. We refrain from formulating these.)

Proof: (a) By assumption, there exists $\{s_n\}_{n\in\mathbb{Z}}$ such that $f_n = s_{n-1} \circ d_n + d_{n+1} \circ s_n$. Here: $s_n : C_n \longrightarrow D_{n+1}$.
Recall: $H_n(f)$ is the map $H_n(C) \to H_n(D)$ induced by f.
Let $\underset{\Uparrow\; Z_n(C)}{x} + B_n(C)$ be a residue class in $H_n(C) = \cancel{B_n(C)/Z_n(C)}$. $Z_n(C)/B_n(C)$. Compute $f_n(x) \in H_n(D)$:

$$x \longmapsto f_n(x) = s_{n-1}(\underbrace{d_n(x)}_{=0:\, x \in Z_n}) + d_{n+1}(s_n(x)) = d_{n+1}(s_n(x)) \in B_n(C) \; (= \text{image of } d)$$

$\Rightarrow$ the residue class of ~~th~~ $f_n(x)$ in $H_n(D)$ vanishes $\Rightarrow H_n(f) = 0$.
(b) By (a), $H_n(f-g) = 0$ $\forall n \in \mathbb{Z}$. $H_n(f-g)$ is the map induced by $f-g$ on n-th homology. This equals $H_n(f) - H_n(g)$, the difference of the maps induced by f and by g on homology.
$\Rightarrow H_n(f) = H_n(g)$. $\square$

When $f: C \to D$ is null-homotopic by s and $g: C \to D$ is null-homotopic by t, then $f+g$ is null-homotopic by $s+t$. This implies that being homotopic is an equivalence relation on $\mathrm{Hom}_{Ch(R\text{-Mod})}(C,D)$.
This is compatible with compositions: Given $u: C' \to C$, $f: C \to D$ and $v: D \to D'$ with f null-homotopic, then $v \circ f \circ u$ is null-homotopic, too. why?

6.12 Definition: The homotopy category $\mathcal{K}(R\text{-Mod})$ of chain complexes is the category with:

objects = objects in $Ch(R\text{-Mod})$

morphisms = residue classes of morphisms in $Ch(R\text{-Mod})$ modulo being homotopic.

$\mathcal{K}(R\text{-Mod})$ is an additive category, but in general not an abelian category. It has a different, also quite interesting structure – it is a triangulated category.

Now we are ready to precisely compare two projective resolutions of the same object or two injective resolutions of the same object. The following result is more general.

6.13 Comparison Theorem: Let X and Y be R-modules and $f: X \to Y$ a homomorphism.
(a) Fix a projective resolution $P_* \xrightarrow{\varepsilon} X$ and a projective resolution $Q_* \xrightarrow{\eta} Y$.
Then there exists a chain map $f_*: P_* \to Q_*$ lifting f, which means

$$\begin{array}{ccccccccccc}
\cdots & \to & P_2 & \to & P_1 & \to & P_0 & \xrightarrow{\varepsilon} & X & \to & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \to & Q_2 & \to & Q_1 & \to & Q_0 & \to & Y & \to & 0
\end{array}$$

is a commutative diagram.

The chain map f_* is unique up to chain homotopy equivalence.
In particular, a projective resolution of X is unique up to homotopy equivalence.

(b) Fix injective resolutions $X \to E^*$ and $Y \to I^*$. Then there exists a cochain map $f^*: E^* \to I^*$ lifting f, which means

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "f"] & E^0 \arrow[r] \arrow[d, "f^0"] & E^1 \arrow[r] \arrow[d, "f^1"] & E^2 \arrow[r] \arrow[d, "f^2"] & \cdots \\ 0 \arrow[r] & Y \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

is a commutative diagram

The cochain map f^* is unique up to cochain homotopy equivalence. In particular, an injective resolution of X is unique up to homotopy equivalence.

In the homotopy category of chain complexes, $K(R\text{-Mod})$, all projective resolutions of X are isomorphic to each other. In the homotopy category of cochain complexes, all injective resolutions of X are isomorphic to each other.

Proof of (b): The cochain map f^* is constructed inductively, using $I^0, I^1, \ldots$ being injective. Recall from 2.8: I injective $\Leftrightarrow$ $\forall$

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[dl, dashed, "\exists"] \\ & I & \end{tikzcd}$$

Existence of f^0:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "f"] & E^0 \arrow[ddl, dashed, "\exists f^0"] \\ & Y \arrow[d] & \\ & I^0 \text{ injective} & \end{tikzcd} \Rightarrow \begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & E^0 \arrow[d, "f^0"] \\ 0 \arrow[r] & Y \arrow[r] & I^0 \end{tikzcd}$$

This square can be extended to the cokernels

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & E^0 \arrow[r] \arrow[d, "f^0"] \arrow[dr, phantom, "\circlearrowleft"] & E^0/X \arrow[r] \arrow[d, "\bar{f}^0"] & 0 \\ 0 \arrow[r] & Y \arrow[r] & I^0 \arrow[r] & I^0/Y \arrow[r] & 0 \end{tikzcd}$$

which leads to

$$\begin{tikzcd} 0 \arrow[r] & E^0/X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E^1 \arrow[ddl, dashed, "\exists f^1"] \\ & I^0/Y \arrow[d] & \\ & I^1 & \end{tikzcd}$$

and this can be continued inductively to get f^*.

Uniqueness of f^* up to homotopy equivalence: When both f^* and g^* lift f, then f^*-g^* lifts $f-f=0$. Thus we can assume that $f=0$ and f^* lifts $f=0$. We have to show that f^* is null-homotopic in this situation.

The construction of the homotopy s also goes inductively, using injectivity of the I^j. Want:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "e^{-1}"] \arrow[d, "0"] \arrow[dl, "s^{-1}"'] & E^0 \arrow[r, "e^0"] \arrow[d, "f^0"] \arrow[dl, "s^0"'] & E^1 \arrow[r, "e^1"] \arrow[d, "f^1"] \arrow[dl, "s^1"'] & E^2 \arrow[r] \arrow[d, "f^2"] \arrow[dl, "s^2"'] & - \\
0 \arrow[r] & Y \arrow[r, "d^{-1}"'] & I^0 \arrow[r, "d^0"'] & I^1 \arrow[r, "d^1"'] & I^2 \arrow[r] & -
\end{tikzcd}$$

Of course, s^{-1} and s^0 can be chosen to be zero.

Existence of $s^1: E^1 \to I^0$: $f^0(e^{-1}(X)) = 0 \Rightarrow e^{-1}(X) \subset \ker(f^0)$

$\Rightarrow 0 \to E^0/e^{-1}(X) \longrightarrow E^1$ and $s^1 \circ e^0 = f^0$, which is what we want.

$$\begin{tikzcd}
0 \arrow[r] & E^0/e^{-1}(X) \arrow[r] \arrow[d, "f_0 - d^{-1}\circ s_0 = f^0"'] & E^1 \arrow[dl, dashed, "\exists s^1"] \\
& I^0 \text{ injective} &
\end{tikzcd}$$

Existence of $s^2: E^2 \to I^1$ such that $s^2 \circ e^1 = f^1 - d^0 \circ s^1$

Now $(f^1 - d^0 \circ s^1)\circ e^0 = f^1 \circ e^0 - d_0 \circ (s^1 \circ e^0) = f^1 \circ e^0 - d^0 \circ (f^0 - d^{-1}\circ s^0) =$

$$= f^1 \circ e^0 - d^0 \circ f^0 + \underbrace{d^0 \circ d^{-1}}_{0} \circ s^0 = f^1 \circ e^0 - d^0 \circ f^0 = 0$$

$\Rightarrow$ as in the case of s^1 there is a factorisation through a quotient:

$$\begin{tikzcd}
0 \arrow[r] & E^1/e^0(E^0) \arrow[r] \arrow[d, "f^1 - d^0\circ s^1"'] & E^2 \arrow[dl, dashed, "\exists s^2"] \\
& I^1 &
\end{tikzcd}$$

And this can be continued inductively.

To compare two injective resolutions of X we choose $X = Y$ and $f = id: X \to X$, and $g = id: X \to X$.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal, "f"'] \arrow[dd, bend right=60, "(id_X)^*"'] & E^0 \arrow[r] \arrow[d, "f^0"] & E^1 \arrow[r] \arrow[d, "f^1"] & E^2 \arrow[r] \arrow[d, "f^2"] & - \arrow[d, "f^*"] \arrow[dd, bend left=60, "g^*\circ f^*"] \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal, "g"'] & I^0 \arrow[r] \arrow[d, "g^0"] & I^1 \arrow[r] \arrow[d, "g^1"] & I^2 \arrow[r] \arrow[d, "g^2"] & - \arrow[d, "g^*"] \\
0 \arrow[r] & X \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & E^2 \arrow[r] & -
\end{tikzcd}$$

Both $(id_X)^*$ (identity everywhere) and $g^* \circ f^*$ lift id_X and therefore must be homotopic. Similarly, $(id_Y)^*$ and $f^* \circ g^*$ are homotopic.
Thus f^* and g^* are homotopy equivalences relating E^* and I^*. $\square$

The construction of the liftings f_i or f^* uses and extends the lifting property of projective and injective modules, respectively. For complexes, whose terms are neither injective nor projective, there are no such lifting results. Our proof of 6.13 does, however, not fully use the assumptions: Only for one complex it is used that the terms are projective or injective, respectively, and exactness also is used in one case only.

The main message the Comparison Theorem has for us is that we should feel free to choose projective or injective resolutions as we find it convenient. In particular, such a choice has no influence when we want to compute homology – which is what we want, as we will see soon.

(Pages 6.15 ff updated February 23rd, 2021: Mix-up of E's and I's corrected in the proof of 6.13, and more details added.)
(Because of the update, the next part of this chapter starts with page 6.17, which it shouldn't any more.)