§6. Complexes and derived functors

Higher extension groups will be defined in the context of universal δ-functors. A natural habitat for these are categories of (co-)chain complexes, which we are going to define now.

6.1 Definition: Let R be a ring and R-Mod the category of left R-modules.
A chain complex C_* of R-modules is a sequence $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules together with a sequence $\{d_n\}_{n\in\mathbb{Z}}$ of R-module homomorphisms $d_n: C_n \to C_{n-1}$ such that $\forall n: d_{n-1}\circ d_n = 0$. (For short: $d^2=0$.)
The maps $d=d_n$ are called differentials.
A cochain complex C^* of R-modules is a sequence $\{C^n\}_{n\in\mathbb{Z}}$ of R-modules together with a sequence $\{d^n\}_{n\in\mathbb{Z}}$ of R-module homomorphisms $d^n: C^n \to C^{n+1}$ such that $\forall n: d^{n+1}\circ d^n = 0$. (For short: $d^2=0$.)
The maps $d=d^n$ are called differentials.
When C_* is a chain complex, $Z_n := Z_n(C_*) := \mathrm{Ker}(d_n)$ is called the module of n-cycles and $B_n := B_n(C_*) := \mathrm{Im}(d_{n+1})$ is the module of n-boundaries.
$H_n := H_n(C_*) := Z_n/B_n$ is the n-th homology of ~~the~~ C_*.
When C^* is a cochain complex, $Z^n := Z^n(C^*) := \mathrm{Ker}(d^n)$ is the module of n-cocycles and $B^n := B^n(C^*) := \mathrm{Im}(d^{n-1})$ is the module of n-coboundaries.
$H^n := H^n(C^*) := Z^n/B^n$ is the n-th cohomology of C^*.

The sloppy notation $d^2=0$ always will mean that all compositions of differentials that are possible have to vanish.
C_* looks like $\dots \to C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{d} C_{-1} \xrightarrow{d} C_{-2} \xrightarrow{d} \dots$ with R-modules C_n, and $d^2=0$ everywhere.
C^* looks like $\dots \to C^{-2} \xrightarrow{d} C^{-1} \xrightarrow{d} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \to \dots$ with R-modules C^n, and $d^2=0$ everywhere.
The only difference is the indexing. This makes sense, since in algebra, geometry, analysis, complexes occur naturally either as chain or cochain complexes, where the indices have a meaning.

Since $d^2=0$, the image B of a differential always is contained in the kernel Z of the next differential.
$H_n = 0$ means at place n, C_* is exact.
$H_n = 0 \ \forall n \in \mathbb{Z}$ means, C_* is a long exact sequence.
Let $Q := \ldots \to 2 \to 1 \to 0 \to -1 \to -2 \to \ldots$ be a quiver.
Then a chain complex C_* is a functor $Q \to R\text{-Mod}$, that is a representation of Q in R-Mod (instead of $\mathbb{K}$-Vect as usual), satisfying the additional condition (relation) $d^2=0$.

To form a category of chain or cochain complexes, we need to define morphisms; in a similar way to morphisms between quiver representations.
6.2 Definition: Let C_* and D_* be chain complexes. A morphism of chain complexes $f_*: C_* \to D_*$ is a sequence of R-module homomorphisms $\{f_n: C_n \to D_n\}_{n\in\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$ the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow{d'_{n-1}} & D_{n-1} \end{array}$$

commutes, where d is the differential of C_* and d' is the differential of D_*.

Morphisms of cochain complexes are defined analogously.
$f_*: C_* \to D_*$ is called a quasi-isomorphism (qis) if and only if it induces isomorphisms $H_n(C_*) \to H_n(D_*)$ $\forall n \in \mathbb{Z}$.
The same term is used for cochain complexes, with induced isomorphisms $H^n(C^*) \to H^n(D^*)$ $\forall n \in \mathbb{Z}$.
Check that there are induced morphisms $Z_n(C_*) \to Z_n(D_*)$ and $B_n(C_*) \to B_n(D_*)$ and thus $H_n(C_*) \to H_n(D_*)$, for instance:

$$\begin{array}{ccccc} Z_n & \to & C_n & \xrightarrow{d} & C_{n-1} \\ \downarrow & & {\scriptstyle f}\downarrow & ? & \downarrow{\scriptstyle f} \\ Z_n & \to & D_n & \xrightarrow[d]{} & D_{n-1} \end{array}$$

Here is a useful example of a quasi-isomorphism: Let M be an R-module and choose a projective resolution $\dots \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \xrightarrow{\pi} M \to 0$

We write M as a chain complex

$$M_* \quad \dots \to 0 \to 0 \to 0 \to M \to 0 \to \dots \qquad \text{all } d=0,\ M \text{ in "degree" } 0$$

and the projective resolution without M as well

$$P_* \quad \dots \to P_n \to \dots \to P_1 \to P_0 \to 0 \to \dots$$ (some maps as in the resolution, P_0 in degree 0, P_n in degree n — here the indexing of chain complexes fits nicely).

There is a morphism of chain complexes

$$\begin{array}{lcccccccc} P_* & \dots \to P_n \to & \dots \to & P_2 \to & P_1 \to & P_0 & \to 0 \to & \dots \\ \pi_* & \dots 0 \downarrow & & 0\downarrow & 0\downarrow & \downarrow \pi & \downarrow 0 & \dots \\ M_* & \dots \to 0 \to & \dots \to & 0 \to & 0 \to & M & \to 0 \to & \dots \end{array}$$

Is it a morphism?

$H_0(M_*) = M$, $H_n(M_*) = 0$ for $n \neq 0$

$H_0(P_*) = M$, $H_n(P_*) = 0$ for $n \neq 0$

π_* is a quasi-isomorphism

(But π_* is not an isomorphism in general, it may not even be possible to define a non-zero morphism $M_* \to P_*$.)

Here, we can choose P_* as we like, any projective resolution of M will do, and all of these complexes then are quasi-isomorphic to M.

There is of course a zero morphism $0_*: C_* \to D_*$, an identity morphism $1_{C}: C_* \to C_*$ and composition is additive. And analogously for cochain complexes.

6.3 Definition: $Ch(R\text{-Mod})$ is the category of chain complexes of R-modules. $CoCh(R\text{-Mod})$ is the category of cochain complexes of R-modules.

6.4 Proposition: $Ch(R\text{-Mod})$ and $CoCh(R\text{-Mod})$ are abelian categories.

Proof (for chain complexes): $\dots \to 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \to \dots$ is the zero object.

For $f_*: C_* \to D_*$ we define $\mathrm{Ker}(f_*)$ as complex

$$K_* \quad \dots \to K_n \to \dots \to K_2 \to K_1 \to K_0 \to K_{-1} \to \dots$$

with $K_n := \mathrm{Ker}(f_n)$. How should the differentials on K_* look like?

$K_* \longrightarrow C_* \longrightarrow D_*$ requires the inclusion of K_* into C_* to be a map of complexes $\Rightarrow$ d_{K_*} of K_* has to be the restriction of d_{C_*}:

$$\begin{array}{ccccccc} \sim \to & K_1 & \xrightarrow{d_1} & K_0 & \xrightarrow{d_0} & K_{-1} & \to \cdots \\ & \downarrow{\scriptstyle incl} & & \downarrow{\scriptstyle incl} & & \downarrow{\scriptstyle incl} & \\ \sim \to & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & C_{-1} & \to \sim \end{array}$$

Check that K_* satisfies the universal property of a kernel.

Similarly, cokernels and hence images are defined termwise and all axioms of abelian categories are satisfied. $\square$

In particular, a sequences $0_* \to C_* \to D_* \to E_* \to 0$ of cochain complexes is exact $\Leftrightarrow$ $\forall n \in \mathbb{Z}$: $0 \to C_n \to D_n \to E_n \to 0$ is exact.

For each n, $C_* \longmapsto H_n(C_*)$ defines a covariant functor

$$H_n : Ch(R\text{-Mod}) \longrightarrow R\text{-Mod}$$

verify that

and $C^* \longmapsto H^n(C^*)$ also defines a covariant functor

$$H^n : CoCh(R\text{-Mod}) \longrightarrow R\text{-Mod}$$

(note: H^n also is covariant, just the indices are different)

We can write homology as a complex:

$$H_* \quad \sim \xrightarrow{d} H_{n+1} \xrightarrow{d} H_n \xrightarrow{d} H_{n-1} \to \cdots$$

By definition of homology, $d = d_{C_*}$ induces $d_{H_*} = 0$.

This complex tells us, at which places C_* is not exact.

Let $0 \to C_* \to D_* \to E_* \to 0$ be an exact sequence of cochain complexes.

Then all rows are exact (to be checked)

$$\begin{array}{c} \vdots \\ H_{n+1}(C_*) \to H_{n+1}(D_*) \to H_{n+1}(E_*) \\ H_n(C_*) \to H_n(D_*) \to H_n(E_*) \\ H_{n-1}(C_*) \to H_{n-1}(D_*) \to H_{n-1}(E_*) \\ \vdots \end{array}$$

(and similarly for ~~chain~~ cochain complexes, with indices $\begin{matrix} n-1 \\ n \\ n+1 \end{matrix}$)

Are there also 0's somewhere? Not in general!

Are the different rows related? Yes!

Enters: the Snake Lemma, Theorem 1A-3, providing a connecting homomorphism.

We extend

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r, "f_n"] \arrow[d, "d"'] & D_n \arrow[r, "g_n"] \arrow[d, "d"'] & E_n \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] & D_{n-1} \arrow[r] & E_{n-1} \arrow[r] & 0
\end{tikzcd}$$

by kernels and cokernels:

$$\begin{tikzcd}
0 \arrow[r] & Z_n(C) \arrow[r, "f_n"] \arrow[d] & Z_n(D) \arrow[r, "g_n"] \arrow[d] & Z_n(E) \arrow[d] & \\
0 \arrow[r] & C_n \arrow[r, "f_n"] \arrow[d, "d_n"] & D_n \arrow[r, "g_n"] \arrow[d, "d_n"] & E_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & D_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & 0 \\
& C_{n-1}/B_{n-1}(C) \arrow[r, "\bar{f}_{n-1}"] & D_{n-1}/B_{n-1}(D) \arrow[r, "\bar{g}_{n-1}"] & E_{n-1}/B_{n-1}(E) \arrow[r] & 0
\end{tikzcd}$$

The diagram is commutative with exact rows.

But there is no H_n in this diagram, so we have to modify it:

$B_n(C) \subset Z_n(C)$ by definition and $Z_n(C) = \ker(d_n)$. Also $\mathrm{Im}(d_n) = B_{n-1}(C) \subset Z_{n-1}(C)$.

$\Rightarrow$ d_n induces $\bar{d}_n : C_n/B_n(C) \xrightarrow{\bar{d}_n} Z_{n-1}(C)$ with $\ker(\bar{d}_n) = Z_n(C)/B_n(C) = H_n(C)$ and $\mathrm{cok}(\bar{d}_n) = Z_{n-1}(C)/B_{n-1}(C) = H_{n-1}(C)$.

Applying these modifications we get a commutative diagram with exact rows

$$\begin{tikzcd}
& H_n(C) \arrow[r, "H_n(f)"] \arrow[d] & H_n(D) \arrow[r, "H_n(g)"] \arrow[d] & H_n(E) \arrow[d] \arrow[ddddll, rounded corners, "\partial_n"] & \\
& C_n/B_n(C) \arrow[r, "\bar{f}_n"] \arrow[d, "\bar{d}_n"] & D_n/B_n(D) \arrow[r, "\bar{g}_n"] \arrow[d, "\bar{d}_n"] & E_n/B_n(E) \arrow[r] \arrow[d, "\bar{d}_n"] & 0 \\
0 \arrow[r] & Z_{n-1}(C) \arrow[r] \arrow[d] & Z_{n-1}(D) \arrow[r] \arrow[d] & Z_{n-1}(E) \arrow[d] & \\
& H_{n-1}(C) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(D) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(E) &
\end{tikzcd}$$

∂_n contributed by the Snake Lemma

$\Rightarrow$ Adding the connecting homomorphisms ∂_n into the picture, we get an infinite sequence — which is exact and contains all $H_n(C_*)$, $H_n(D_*)$ and $H_n(E_*)$. This proves:

6.5 Theorem: Let $0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0$ be a short exact sequence of chain complexes. Then there exists a long exact sequence

$$\dots \to H_{n+1}(E_*) \xrightarrow{\partial_{n+1}} H_n(C_*) \xrightarrow{H_n(f)} H_n(D_*) \xrightarrow{H_n(g)} H_n(E_*) \xrightarrow{\partial_n} H_{n-1}(C) \to \dots,$$

where ∂_n is the connecting homomorphism.

An analogous assertion holds true for cochain complexes:

$$\dots \to H^{n-1}(E^*) \xrightarrow{\partial_{n-1}} H^n(C^*) \to H^n(D^*) \to H^n(E^*) \xrightarrow{\partial^n} H^{n+1}(C^*) \to \dots$$

The two sequences are called <u>long exact homology sequence</u> and <u>long exact cohomology sequence.</u>

One can write the sequences also in the form

$$\begin{tikzcd} H_*(C_*) \arrow[rr, "H_*(f)"] & & H_*(D_*) \arrow[dl, "H_*(g)"] \\ & H_*(E_*) \arrow[ul, squiggly, "\partial"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H^*(C^*) \arrow[rr, "H^*(f)"] & & H^*(D^*) \arrow[dl] \\ & H^*(E^*) \arrow[ul, squiggly, "\partial"] & \end{tikzcd}$$

where $\nwarrow$ indicates that here a change of degree happens (downwards from $n+1$ to n or upwards from n to $n+1$).

One can show that assigning a long exact sequence to a short exact sequence of complexes in this way is functorial. The functor starts in a category whose objects are such ses and the target category has objects that are long exact sequences.

Long exact (co-)homology sequences are very useful tools. In applications one rarely has to know the connecting homomorphism in detail. More often it helps just to know that some H_n or H^n are zero, which then provides information about other data.