Appendix to Chapter 6: Proof of Theorem 6.17

We only consider the case of $F: \mathcal{A} \to \mathcal{B}$ being right exact, assuming that $\mathcal{A}$ has enough projective objects. We have to show that the left derived functors $L_i F$ form a homological δ-functor and satisfy universality. The proof for F left exact and its right derived functors $R^i F$ would proceed in a very similar way.

Claim 1: The $L_i F$ are well-defined.

Proof: For A in $\mathcal{A}$, let P_* and Q_* be two projective resolutions of A. By the Comparison Theorem 6.13, the identity on A, $1_A: A \to A$, can be lifted to $f_*: P_* \to Q_*$ and to $g_*: Q_* \to P_*$. Then $id_*: P_* \to P_*$ and $g_* \circ f_*: P_* \to P_*$ both are lifts, too, and similarly $id_*, f_* \circ g_*: Q_* \to Q_*$. By 6.13, lifts of the same map are homotopy equivalent. Lemma 6.13 implies that $H(f_*)$ and $H(g_*)$ are isomorphisms on homology and $H(f_*) = H(g_*)$. Therefore, the $L_i F$ do not depend on the choice of projective resolution.

Claim 2: The $L_i F$ are additive functors.

Proof: F and (co)homology are additive functors. Let $f: A_1 \to A_2$ be a morphism in $\mathcal{A}$. A lift $f_*: P_*(A_1) \to P_*(A_2)$ is unique up to homotopy, by 6.13 (so addition of two lifts is unique only up to homotopy). The induced morphism in homology $H_i(F(P_*(A_1))) \to H_i(F(P_*(A_2)))$ is independent of the chosen lifting of f. Thus, on this level, addition of morphism is well-defined.

Claim 3: The $L_i F$ satisfy condition (1) in Definition 6.15.

Proof: Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. We choose projective resolutions $P_* \to A$ and $Q_* \to C$.

The Horseshoe Lemma 6.9 allows to choose a projective resolution $(P\oplus Q)_* \to B$ that fits into an exact sequence of complexes $0 \to P_* \to (P\oplus Q)_* \to Q_* \to 0$, split in each degree and lifting the given ses $0 \to A \to B \to C \to 0$.
F is by assumption right exact, but not necessarily exact. Like any additive functor F is, however, exact on split exact sequences, such as $0 \to P_n \to P_n\oplus Q_n \to Q_n \to 0$
$\Rightarrow 0 \to F(P_*) \to F((P\oplus Q)_*) \to F(Q_*) \to 0$ is a short exact sequence of chain complexes, and we can apply Theorem 6.5 to get a long exact homology sequence with connecting homomorphisms coming from the Snake Lemma. Hence, condition (1) is satisfied.

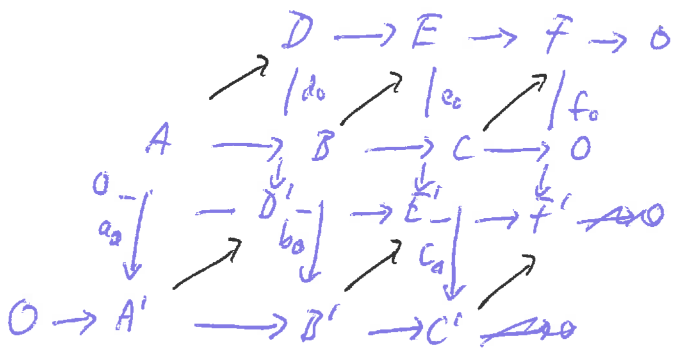
Checking condition (2) requires much more work. We prepare for that by checking naturality of the connecting homomorphism in the Snake Lemma.

<u>Claim 4:</u> The connecting homomorphism ∂ is natural.

This means: Given are two situations to which the Snake Lemma (Theorem 1A.3) can be applied:

$$\begin{tikzcd} & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & D \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D' \arrow[r] & E' \arrow[r] & F' & \end{tikzcd}$$

and there are morphisms (in black) of exact sequences relating the two situations



Viewing this diagram in 3-dimensional space, we have to apply the Snake Lemma twice, once to the front and once to the back, and then compare, which means checking commutativity of the diagram relating the two long exact sequences:

$$\begin{tikzcd}
\text{front:} & \mathrm{Ker}(a_0) \arrow[r] \arrow[d] & \mathrm{Ker}(b_0) \arrow[r] \arrow[d] & \mathrm{Ker}(c_0) \arrow[r, "\delta"] \arrow[d] & \mathrm{Cok}(a_0) \arrow[r] \arrow[d] & \mathrm{Cok}(b_0) \arrow[r] \arrow[d] & \mathrm{Cok}(c_0) \arrow[d] \\
\text{back:} & \mathrm{Ker}(d_0) \arrow[r] & \mathrm{Ker}(e_0) \arrow[r] & \mathrm{Ker}(f_0) \arrow[r, "\delta'"] & \mathrm{Cok}(d_0) \arrow[r] & \mathrm{Cok}(e_0) \arrow[r] & \mathrm{Cok}(f_0)
\end{tikzcd}$$

Commutativity of the squares not involving δ or δ' is checked by diagram chasing.
Commutativity of the squares involving δ and δ' is checked by following the construction of the connecting homomorphism.
This proves Claim 4.

Now the hard part starts.
Claim 5: The L_iF satisfy Condition (2) in Definition 6.15.
This means: We are given a morphism of short exact sequences in $\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r] \arrow[d, "h"'] & C' \arrow[r] \arrow[d, "g"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd} \quad (*)$$

and we have to verify commutativity of

$$\begin{tikzcd}
T_n(C') \arrow[r, "\delta_n'"] \arrow[d, "T_n(f)"'] & T_{n-1}(A') \arrow[d, "T(g)"] \\
T_n(C) \arrow[r, "\delta_n"] & T_{n-1}(A)
\end{tikzcd} \quad \text{in } \mathcal{B} \quad \forall n \in \mathbb{Z}$$

Proof: We have to choose projective resolutions of all six terms in $(*)$, lift the given maps to maps between chain complexes and then apply F and then homology. Both F and taking homology are functors are functors, and homology uses the connecting homomorphism coming from the Snake Lemma. By claim 4, the connecting homomorphism is natural. But we have to make sure that all the resolutions and lifts we choose are compatible so that Claim 4 can be applied and the naturality of the connecting homomorphism can be used. This means precisely the following:
Given $0 \to A \to B \to C \to 0$, the Horseshoe Lemma 6.9 tells us how to choose projective resolutions $P(A)_*$, $P(B)_*$ and $P(C)_*$ such that the given maps can be lifted to an ses of chain complexes $0 \to P(A)_* \to P(B)_* \to P(C)_* \to 0$ and for $0 \to A' \to B' \to C' \to 0$ we can choose $Q(A')_*$, $Q(B')_*$ and $Q(C')_*$. What we still have to do is to lift the maps (f,g,h) & in $(*)$ so that

$$(+) \quad \begin{tikzcd}
0 \arrow[r] & Q(A')_* \arrow[r] \arrow[d, "\hat{f}"'] & Q(B')_* \arrow[r] \arrow[d, "\hat{h}"'] & Q(C')_* \arrow[r] \arrow[d, "\hat{g}"'] & 0 \\
0 \arrow[r] & P(A)_* \arrow[r] & P(B)_* \arrow[r] & P(C)_* \arrow[r] & 0
\end{tikzcd}$$

commutes.
Then we are done.

The Comparison Theorem 6.13 tells us that we can construct each of the lifts $\hat{f}, \hat{g}$ and $\hat{h}$, separately. But 6.13 does not guarantee commutativity of (+). Therefore, we can take $\hat{f}$ and $\hat{g}$ as given and construct new $\hat{h}$ making the diagram (+) commutative.

<u>Construction of $\hat{h}$:</u>

By the construction in the Horseshoe Lemma 6.9, for each n, $Q(B')_n = Q(A')_n \oplus Q(C')_n$ and $P(B)_n = P(A)_n \oplus P(C)_n$. So, h_n: $\begin{matrix} Q(A')_n \\ \oplus \\ Q(C')_n \end{matrix} \longrightarrow \begin{matrix} P(A)_n \\ \oplus \\ P(C)_n \end{matrix}$ can be written as a 2×2-matrix with entries in
$Q(A')_n \to P(A)_n$, etc. In order to make (+) commutative, we must choose the diagonal entries as $f_n: Q(A')_n \longrightarrow P(A)_n$ and $g_n: Q(C')_n \to P(C)_n$, respectively, and $Q(A')_n \to P(C)_n$ has to be zero. But the fourth entry, $\mu_n: Q(C')_n \to P(A)_n$ is not yet determined. So we have to find μ_n, and the whole family $\hat{h} = (h_n)_n$ then has to satisfy:
- $\hat{h}$ has to be a lift of $h: B' \to B$
- $\hat{h}$ has to be a morphism of complexes, i.e. satisfy $d \circ \hat{h} = \hat{h} \circ d$.

Of course, the first condition is only relevant in the first step of the inductive construction of h_n, which will follow now, and which will produce a morphism of complexes.

Induction start, $n=0$:

$Q(C')_0$ and $P(A)_0$ are the rightmost non-zero terms in the complexes $Q(C')_*$ and $P(A)_*$.
$\Rightarrow$ The differentials starting in these terms are zero and $d \circ h_0 = h_0 \circ d$ is for free.
The relevant condition to be fulfilled is that $\hat{h}$ lifts h. This means

$$\begin{tikzcd} Q(A')_0 \oplus Q(C')_0 \arrow[r, "\varepsilon"] \arrow[d, "h_0"] & B' \arrow[d, "h"] \\ P(A)_0 \oplus P(C)_0 \arrow[r, "\eta"] & B \end{tikzcd}$$
has to commute

$\hat{f}$ lifts $f \Rightarrow$
$$\begin{tikzcd} Q(A')_0 \arrow[r] \arrow[d, "f_0"] & B' \arrow[d, "h \text{ (restricted to } A' \to A)"] \\ P(A)_0 \arrow[r] & B \end{tikzcd}$$
commutes

$\hat{g}$ lifts $g \Rightarrow$
$$\begin{tikzcd} Q(C')_0 \arrow[r] \arrow[d, "g_0"] & C' \arrow[d, "g"] \\ P(C)_0 \arrow[r] & C \simeq B/A \end{tikzcd}$$
commutes as well $\Rightarrow$
$$\begin{tikzcd} Q(C')_0 \arrow[r, "\varepsilon''"] \arrow[d, "g_0"] & B' \arrow[d, "h"] & \\ P(C)_0 \arrow[r, "\eta''"] & B \arrow[r, two heads] & C = B/A \end{tikzcd}$$
commutes when taking residue classes in C

But this diagram only commutes after taking residue classes in $B/A = C$.
The map $\beta := h\circ\varepsilon'' - \eta''\circ g_0 : Q(C')_0 \to B$ would be zero, if the diagram would commute (which is not true in general). In other words, composing β with the quotient map $B \to B/A$ is zero and hence the image of β is contained in A.
This will tell us how to construct ψ_0.
What we really want is

$$\begin{tikzcd} Q(C')_0 \arrow[r, "\varepsilon''"] \arrow[d, "(\psi_0, g_0)"'] & B' \arrow[d, "h"] \\ P(A)_0 \oplus P(C)_0 \arrow[r, "\mu''"] & B \end{tikzcd}$$

to commute. This means exactly $h\circ\varepsilon'' = \mu''\circ\psi_0 + \eta''\circ g_0$ which is equivalent to $\eta''\circ\psi_0 = \beta$ $(= h\circ\varepsilon'' - \eta''\circ g_0)$ and this is equivalent to

$$\begin{tikzcd} & Q(C')_0 \arrow[dl, "\psi_0"'] \arrow[d, "\beta"] \\ P(A)_0 \arrow[r] & B \end{tikzcd}$$

to commute (for ψ_0 still to be chosen). $\beta : Q(C')_0 \to B$ has image in B, and $P(A)_0 \to B$ is given by the surjective map $P(A)_0 \twoheadrightarrow A \subset B$.
Since $Q(C')_0$ is projective, there exists $\psi_0 : Q(C')_0 \to P(A)_0$ making the diagram commutative.
This finishes the induction start. The condition about lifting h is now settled generally, provided we find $\hat h$ and show it is a morphism of complexes.

Induction step for $n \geq 1$:
We have to find ψ_n such that $d\circ\hat h = \hat h\circ d$ (at place n). ψ_{n-1} is already defined by induction.
Now, differentials do not vanish any more and need to be taken into account.
We fix notation: As before, $h : \begin{matrix} Q(A') \\ \oplus \\ Q(C') \end{matrix} \longrightarrow \begin{matrix} P(A) \\ \oplus \\ P(C) \end{matrix}$ is given by a matrix $\begin{pmatrix} f & \kappa \\ \cancel{\psi}\,0 & g \end{pmatrix}$ (all with index n)
which acts by left multiplication on column vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \begin{matrix} Q(A') \\ \oplus \\ Q(C') \end{matrix}$. Similarly, differentials (whose existence has been shown in the Horseshoe Lemma) are given by matrices $d_Q = \begin{pmatrix} d_{A'} & \lambda_Q \\ 0 & d_{C'} \end{pmatrix}$ and $d_P = \begin{pmatrix} d_A & \lambda_P \\ 0 & d_C \end{pmatrix}$ (again with index n, the 0 in the corner coming from commutativity of the diagrams belonging to the ses of complexes $0 \to Q(A')_* \to Q(B')_* \to Q(C')_* \to 0$ and $0 \to P(A)_* \to P(B)_* \to P(C)_* \to 0$, where $d_{A'}, d_{C'}$ and d_A, d_C are given differentials).

The condition to be satisfied is $\hat{h} \circ d \overset{!}{=} d \circ \hat{h}$, which means $\hat{h}$ is a morphism of complexes. In terms of matrices
$$\underbrace{\begin{pmatrix} f & \mu \\ 0 & g \end{pmatrix}}_{\text{indices } n-1} \begin{pmatrix} d_{A'} & d_Q \\ 0 & d_{C'} \end{pmatrix} \overset{!}{=} \begin{pmatrix} d_A & d_P \\ 0 & d_C \end{pmatrix} \underbrace{\begin{pmatrix} f & \mu \\ 0 & g \end{pmatrix}}_{\text{indices } n}$$

Since $\hat{f} \circ d = d \circ \hat{f}$ and $\hat{g} \circ d = d \circ \hat{g}$, equalities in the diagonal entries are automatic, and there is just one equation to be satisfied, in the right upper corner:
$$d \circ \mu_n \overset{!}{=} \underbrace{\mu_{n-1} \circ d + f \circ \lambda - \lambda \circ g}_{\text{defined already by induction}} : Q(C')_n \longrightarrow P(A)_{n-1}$$
and, by induction, μ_{n-1} satisfies the same condition, i.e. ~~μ_{n-1}~~
$$d \circ \mu_{n-1} = \mu_{n-2} \circ d + f \circ \lambda - \lambda \circ g \quad \text{(different indices from above)}$$
We need μ_n such that
$$\begin{tikzcd} & Q(C')_n \arrow[dl, dashed, "\mu_n"'] \arrow[d, "\mu_{n-1} \circ d + f \circ \lambda - \lambda \circ g"] \\ P(A)_n \arrow[r, "d"] & P(A)_{n-1} \end{tikzcd}$$
commutes. This is now similar to the last step in the induction start:

The image of $d : P(A)_n \longrightarrow P(A)_{n-1}$ is, by exactness of the resolution, the kernel of the d starting at $P(A)_{n-1}$. Thus we will show that the image of $\mu_{n-1} \circ d + f \circ \lambda - \lambda \circ g$ is contained in $\mathrm{Ker}(d) \subset P(A)_{n-1}$. Then $Q(C')_n$ being projective implies that the lift μ_n exists, and the induction step is done.

It remains to check that $d \circ (\mu_{n-1} \circ d + f \circ \lambda - \lambda \circ g) = 0$:
$$d \circ (\mu_{n-1} \circ d + f \circ \lambda - \lambda \circ g) = (d \circ \mu_{n-1}) \circ d + d \circ f \circ \lambda - d \circ \lambda \circ g$$
$$= (\mu_{n-1} \circ \underbrace{d \circ d}_{0} + f \circ \lambda \circ d - \lambda \circ g \circ d) + d \circ f \circ \lambda - d \circ \lambda \circ g$$
We check $d \circ f \circ \lambda = - f \circ \lambda \circ d$ and also $d \circ \lambda \circ g = - \lambda \circ g \circ d$:

Since f is a chain map, $d \circ f = f \circ d$, and for the same reason, $d \circ g = g \circ d$.

Since $\begin{pmatrix} d & \lambda \\ 0 & d \end{pmatrix}$ is a differential, $0 = \begin{pmatrix} d & \lambda \\ 0 & d \end{pmatrix}\begin{pmatrix} d & \lambda \\ 0 & d \end{pmatrix} \Rightarrow d \circ \lambda + \lambda \circ d = 0$. And that is what we need.

This finishes the proof of claim 5. Now we know that the $L_i F$ form a homological δ-functor. What remains to be shown is that they form a universal homological δ-functor. This is the final claim.

Claim 6: The homological δ-functor formed by the L_iF is universal.

This means: Let T_* be a homological δ-functor and $\varphi_0: T_0 \to F$ a morphism of functors, i.e. a natural transformation. We have to show: There exists a unique extension to a morphism $\varphi: T_* \to L_*F$ of δ-functors.

Proof: We proceed by induction. The induction start is the assumption $\exists \varphi_0: T_0 \to F$. Suppose for a given $n>0$: $\varphi_i: T_i \to L_iF$ are already defined for $0 \le i < n$ and satisfy all conditions. That is, they are natural transformations and commute with the appropriate δ_i's as required.

For A in $\mathcal{A}$ we have to find a morphism $T_n(A) \xrightarrow{\varphi_n(A)} L_nF(A)$ which has the required properties. We will shortly see that there is exactly one candidate $\varphi_n(A)$.

Choose an exact sequence $0 \to K \to P \to A \to 0$ with P projective. This exists by assumption on $\mathcal{A}$. Since P is projective, it has a projective resolution $\dots \to 0 \to 0 \to \dots \to 0 \to P \to 0$. Therefore, $L_nF(P) = 0$ for $n \ge 1$. $\Rightarrow$ There is a commutative diagram with exact rows

$$\begin{tikzcd}
 & T_n(A) \arrow[r, "\delta_n"] & T_{n-1}(K) \arrow[r, "\delta_{n-1}"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(P) \arrow[d, "\varphi_{n-1}"] \\
0 \arrow[r] \arrow[d, equal] & L_nF(A) \arrow[r, "\delta_n"] & L_{n-1}F(K) \arrow[r, "\delta_{n-1}"] & L_{n-1}F(P) \\
L_nF(P) & & &
\end{tikzcd}$$

Existence of $\varphi_n(A): T_n(A) \to L_nF(A)$ commuting with δ_n follows by diagram chasing:

$$\begin{tikzcd}
T_n(A) \ni a \arrow[r, mapsto] & \delta_n(a) \arrow[r, mapsto] \arrow[d] & \delta_{n-1}(\delta_n(a)) = 0 \arrow[d] \\
 & \varphi_{n-1}(\delta_n(a)) \arrow[r, mapsto] & 0
\end{tikzcd}$$

and there is no choice $\Rightarrow$ $\varphi_n(A)$ is unique

$$L_nF(A) \cong \mathrm{Im}(\delta_n) = \mathrm{Ker}(\delta_{n-1})$$

We still have to show that φ_n is a natural transformation and that it commutes with all δ_n's, not only those coming from the chosen sequence.

First we show that φ_n is a natural transformation. Let $f: A' \to A$ be a morphism in $\mathcal{A}$ and $0 \to K' \to P' \to A' \to 0$ an exact sequence with P' projective.

$$\rightsquigarrow \quad \begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, "h"] & P' \arrow[r] \arrow[d, "\exists g"] & A' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

P' projective $\Rightarrow$ $\exists$ lift g of f, and g induces $h: K' \to K$ such that the diagram is commutative.

Consider the following diagram:

$$\begin{tikzcd}
T_n(A') \arrow[rrr, "T_n(f)"] \arrow[dddd, "\varphi_n(A')"'] \arrow[dr, "\delta"'] & & & T_n(A) \arrow[dddd, "\varphi_n(A)"] \arrow[dl, "\delta"] \\
& T_{n-1}(K') \arrow[r, "T_{n-1}(h)"] \arrow[d, "\varphi_{n-1}(K')"'] & T_{n-1}(K) \arrow[d, "\varphi_{n-1}(K)"] & \\
& L_{n-1}F(K') \arrow[r, "L_{n-1}F(h)"'] & L_{n-1}F(K) & \\
& & & \\
L_nF(A') \arrow[rrr, "L_nF(f)"'] \arrow[uur, "\delta"] & & & L_nF(A) \arrow[uul, "\delta"']
\end{tikzcd}$$

The inner square commutes by induction. By naturality of the connecting homomorphism, all the small quadrilaterals (each involving two connecting homomorphisms) also commute. This implies

$$\delta \circ L_nF(f) \circ \varphi_n(A') = \delta \circ \varphi_n(A) \circ T_n(f)$$

Since P and P' are projective, $\delta : L_nF(A') \longrightarrow L_{n-1}F(K')$ and, in particular, $\delta : L_nF(A) \longrightarrow L_{n-1}F(K)$ are monomorphisms $\Rightarrow$ δ can be cancelled in the equation above and $L_nF(f) \circ \varphi_n(A') = \varphi_n(A) \circ T_n(f)$, which means φ_n is a natural transformation. (Choosing $A = A'$, $f = id$, but P and P' different, this also tells us that $\varphi_n(A)$ is the same for both choices, i.e. $\varphi_n(A)$ does not depend on the choice of P.)

The final step is to verify that φ_n commutes with δ_n. For a given sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ this means we have to show that

$$\begin{tikzcd}
T_n(C) \arrow[r, "\delta"] \arrow[d, "\varphi_n"'] & T_{n-1}(A) \arrow[d, "\varphi_{n-1}"] \\
L_nF(C) \arrow[r, "\delta"] & L_{n-1}F(A)
\end{tikzcd}$$

is commutative

To show this, we choose another sequence $0 \to K \to P \to C \to 0$ with P projective in $\mathcal{A}$ and compare the sequences

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, dashed, "g"] & P \arrow[r] \arrow[d, dashed, "f"] & C \arrow[r] \arrow[d, "id_C"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

f exists since P is projective,
g is induced $\Rightarrow$ the diagram commutes

The previous commutative diagram yields a commutative diagram

$$\begin{tikzcd} \arrow[r] & T_n(P) \arrow[r] \arrow[d, "\varphi_n"'] \arrow[dr, phantom, "\circlearrowright"] & T_n(C) \arrow[r, "\delta"] \arrow[d, "\varphi_n"'] \arrow[dr, phantom, "\circlearrowright"] & T_{n-1}(K) \arrow[r] \arrow[d, "\varphi_{n-1}"'] \arrow[dr, phantom, "\circlearrowright"] & T_{n-1}(P) \arrow[r] \arrow[d, "\varphi_{n-1}"'] & {} \\ \arrow[r] & L_nF(P) \arrow[r] & L_nF(C) \arrow[r, "\delta"] & L_{n-1}F(K) \arrow[r] & L_{n-1}F(P) \arrow[r] & {} \end{tikzcd}$$

but also (only using the left hand square and going vertically instead of horizontally) a commutative square

$$\begin{tikzcd} T_{n-1}(K) \arrow[r, "T_n(g)"] \arrow[d, "\varphi_{n-1}"'] \arrow[dr, phantom, "\circlearrowright"] & T_{n-1}(A) \arrow[d, "\varphi_{n-1}"] \\ L_{n-1}F(K) \arrow[r, "L_{n-1}F(g)"] & L_{n-1}F(A) \end{tikzcd}$$

Putting this square on the right hand side of the middle square above yields

$$\begin{tikzcd} T_n(C) \arrow[r, "\delta"] \arrow[d, "\varphi_n"'] \arrow[dr, phantom, "\circlearrowright"] & T_{n-1}(K) \arrow[r, "T_n(g)"] \arrow[d, "\varphi_{n-1}"'] \arrow[dr, phantom, "\circlearrowright"] & T_{n-1}(A) \arrow[d, "\varphi_{n-1}"] \\ L_nF(C) \arrow[r, "\delta"] & L_{n-1}F(K) \arrow[r, "L_{n-1}F(g)"] & L_{n-1}F(A) \end{tikzcd}$$

Again by naturality of δ, the composition $T_n(g) \circ \delta$ in the upper row is $\delta: T_n(C) \longrightarrow T_{n-1}(A)$, and the composition in the bottom row is $\delta: L_nF(C) \longrightarrow L_{n-1}F(A)$.
The outer rectangle commutes $\Rightarrow$ φ_n commutes with δ_n. $\square$