§5. Categories and functors

5.1 Definition: A category $\mathcal{C}$ consists of the following data:

- a class $Ob\,\mathcal{C}$ of objects of $\mathcal{C}$
- for each ordered pair $(X,Y) \in Ob\,\mathcal{C} \times Ob\,\mathcal{C}$ a set of morphisms, $\mathcal{C}(X,Y) =: Hom_{\mathcal{C}}(X,Y)$ (with domain X and target Y) such that there is an associative composition (for each ordered triple (X,Y,Z))

$$\begin{array}{ccccc} Hom_{\mathcal{C}}(X,Y) & \times & Hom_{\mathcal{C}}(Y,Z) & \longrightarrow & Hom_{\mathcal{C}}(X,Z) \\ \Psi & & \Psi & & \Psi \\ f & & g & \longmapsto & h =: g \circ f \end{array}$$

and for each $X \in Ob\,\mathcal{C}$ there is an identity object $id_X = 1_X \in Hom_{\mathcal{C}}(X,X)$ satisfying $1_X \circ f = f$ and $g \circ 1_X = g$ whenever composition is possible.

Note: $Ob\,\mathcal{C}$ need not be a set, and in many examples it isn't. The objects $X, Y, Z, -$ need not be sets either. But for X, Y fixed, $Hom_{\mathcal{C}}(X,Y)$ must be a set, which may be empty unless $X = Y$.

Examples:
- The category $\mathcal{C} = Set$: objects are sets, morphisms are maps between sets, composition as usual.
- A a ring, then A-Mod is the category of left A-modules in the obvious way. It contains A-mod $= A$-mofg, the category of finitely generated left A-modules, as a full subcategory: This means $Ob(A\text{-mod}) \subset Ob(A\text{-Mod})$ and for $X, Y \in Ob(A\text{-mod})$ (we often write: $X, Y \in A$-mod): $Hom_{A\text{-Mod}}(X,Y) = Hom_{A\text{-mod}}(X,Y)$. For $A = K$ a field we get the categories K-Vect and K-vect of (finite dimensional) K-vector spaces.
- ~~An~~ G a group, this is a category with one object $\{G\}$ and $Hom(\{G\},\{G\}) := G$ with multiplication of group elements as composition. $1_{\{G\}}$ is the unit element of G.
- There are categories of topological spaces, of algebraic varieties, and many others, almost everywhere in mathematics.

- Sheaves and generally functors form categories whose objects are, in general, no sets. The category of categories also has objects that are no sets.
- Let Q be a quiver. Then Q is a category with $Ob\, Q = Q_0$, the set of vertices and morphism sets $Q(x,y) = \{\text{paths } x \to \dots \to y\}$, where composition is by concatenation of paths.

In any category $\mathcal{C}$ one can define isomorphism (by existence of an inverse), monomorphism, epimorphism, kernel, cokernel (if 0 is defined), pullback, pushout, product of objects, coproduct of objects, and so on; we know already how to prove uniqueness of, e.g., kernel and cokernel, from the general definition for categories. But we also know that existence needs to be proven for each category, and it may fail.

In representation theory we are usually interested in categories with additional properties.

5.2 Definition: An *additive category* is a category $\mathcal{C}$ satisfying:

- for any finite set $X_1, \dots, X_n \in Ob\,\mathcal{C}$ of (not necessarily pairwise different) objects, the coproduct (= direct sum) $X_1 \oplus \dots \oplus X_n \in Ob\,\mathcal{C}$ exists
- for all $X, Y \in Ob\,\mathcal{C}$ (not necessarily different), $Hom_{\mathcal{C}}(X,Y)$ is an abelian group (the neutral element is denoted by 0_{XY})
- composition of morphisms is bilinear: $(f_1+f_2)\circ g = f_1\circ g + f_2 \circ g$ and $f\circ(g_1+g_2) = f\circ g_1 + f \circ g_2$ (whenever it makes sense)
- there exists a zero object $0 \in Ob\,\mathcal{C}$ satisfying $1_0 = 0_{00} \in Hom_{\mathcal{C}}(0,0)$

A special case is:

5.3 Definition: Let k be a field. A *k-category* $\mathcal{C}$ satisfies that for all X, Y the morphism set $Hom_{\mathcal{C}}(X,Y)$ is a k-vectorspace, and composition of morphisms is k-bilinear.

In additive categories one can, for instance, define kernels and cokernels and show that f is a monomorphism if and only if $\mathrm{Ker}(f)=0$ (both the object $\mathrm{Ker}(f)$ and the morphism $\mathrm{Ker}(f)\to\mathrm{domain}(f)$ are zero).
Existence of kernel or cokernel is not guaranteed in additive categories.

5.4 Definition: An additive category $\mathcal{C}$ is called abelian category if it satisfies: each morphism $f: X\to Y$ has a kernel and a cokernel and the natural morphism $g: \underbrace{\mathrm{coker}(\mathrm{Ker}(f))}_{\text{this is then called the image } \mathrm{Im}(f)} \to \mathrm{Ker}(\mathrm{Coker}(f))$ is an isomorphism.

(The natural morphism is constructed using the universal properties of $\mathrm{Ker}(f)$ and $\mathrm{Cok}(f)$:

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d] \arrow[dr, "\exists!"] & Y \arrow[r, "\pi"] & \mathrm{Cok}(f) \\
& \mathrm{Cok}(\iota) \arrow[r, "\exists! g"] & \mathrm{Ker}(\pi) \arrow[u] &
\end{tikzcd}$$
)

The category Ab of abelian groups is an abelian category – which explains the name. For any ring R, the module category R-Mod is an abelian category. R-mod fails, in general, to be abelian. The problem is that the submodule $\mathrm{Ker}(f)$ of the finitely generated module X need not be finitely generated itself.

In abelian categories one can define exact sequences, extensions and equivalence of extensions. Existence of projective (or injective) (co-)resolutions is, however, not guaranteed.

Objects of an abelian category are, in general, not sets. However, the embedding theorem of Mitchell (aka Freyd–Mitchell embedding theorem) states that every abelian category $\mathcal{A}$ is a full subcategory of R-Mod for some ring R.
(More precisely, $\mathcal{A}$ is equivalent to a full subcategory of R-Mod, that is, there exists a full subcategory $\mathcal{C}$ with $\mathrm{Ob}\,\mathcal{C} \overset{\text{bijection}}{\simeq} \mathrm{Ob}\,\mathcal{A}$ and the same morphisms as in $\mathcal{A}$.)

to be defined soon

Important consequence: In abelian categories one is allowed to use diagram-chasing in proofs. In particular, we can use five lemma, snake lemma etc.

How can we compare or relate two categories (and thus relate, for instance, algebra with geometry or topology or analysis)? And can one define something like an isomorphism between two categories?
We first have to define a morphism between categories:

5.5 Definition: Let $\mathcal{C}$ and $\mathcal{D}$ be categories.
A covariant functor $F: \mathcal{C} \to \mathcal{D}$ consists of a map

$$Ob(\mathcal{C}) \ni X \longmapsto F(X) \in Ob\,\mathcal{D}$$

and maps $Hom_{\mathcal{C}}(X,Y) \ni f \mapsto F(f) \in Hom_{\mathcal{D}}(F(X), F(Y))$ satisfying

$$F(1_X) = 1_{F(X)} \text{ and } F(f \circ g) = F(f) \circ F(g) \text{ for all } X, Y \text{ and all map } f \text{ and } g$$

that can be composed.
A contravariant functor $G: \mathcal{C} \to \mathcal{D}$ consists of a map

$$Ob(\mathcal{C}) \ni X \longmapsto G(X) \in Ob\,\mathcal{D}$$

and maps $Hom_{\mathcal{C}}(X,Y) \ni f \mapsto G(f) \in Hom_{\mathcal{D}}(\overbrace{G(Y), G(X)}^{\text{here is the difference}})$ satisfying

$$G(1_X) = 1_{G(X)} \text{ and } G(f \circ g) = \underbrace{G(g) \circ G(f)}_{\text{again: different}} \text{ for all } X, Y \text{ and all maps } f \text{ and } g$$

that can be composed.

Examples: • $Id: \mathcal{C} \to \mathcal{C}$ with $Id(X) = X$ and $Id(f) = f$ is a covariant functor.
• For $X_0 \in Ob\,\mathcal{D}$, $F: \mathcal{C} \to \mathcal{D}$ sends $X \mapsto X_0$, $f \mapsto 1_{X_0}$ (the constant functor).

Check the axioms and decide co- or contravariant.

• P: Set $\to$ Set sends X to the powerset $P(X)$, covariant and P': Set $\to$ Set sends X to the powerset $P(X)$, contravariant.

Define on morphisms.

• In topology, you may have seen the fundamental group $\pi_1: Top_* \longrightarrow$ Groups.
• Forgetful functor F: Groups $\to$ Set, $G \mapsto G$, $f \mapsto f$ (forgetting structure)

- K a field, $D = \mathrm{Hom}_K(-,K) : K\text{-Vect} \to K\text{-Vect}$ sends V to V^*, f to f^* Check the details, co- or contravariant?
- Q a quiver, seen as a category with objects = vertices, morphisms = paths. A functor $M : Q \to K\text{-Vect}$ (K a field), sends $i \in Q_0$ to a vector space $M(i)$, and a path $p : i \rightsquigarrow j$ to $M(p) : M(i) \to M(j)$, a linear map
  $\Rightarrow$ a functor $M : Q \to K\text{-Vect}$ (or to $K\text{-Vect}$) is exactly a finite dimensional (or infinite dimensional) representation of Q over K, i.e. a right KQ-module
- A ring R is a category (additive, not abelian) with one object A and morphisms $\mathrm{Hom}(A,A) := A$.
  A functor $M : R \to Ab$ (the category of abelian groups) sends R to an abelian group $M(R)$ and a morphism $a \in A$ to $M(a) : M(R) \to M(R)$, a morphism of abelian groups. $1_A \longmapsto M(1_A) = id_{M(R)}$, $a \cdot b \mapsto M(ab) = \underbrace{M(a)M(b)}_{= M(a)\circ M(b)\text{ (our choice)}}$, if M is covariant, or $M(ab) = \underbrace{M(b)M(a)}_{= M(b)\circ M(a)}$, if M is contravariant.

$\Rightarrow$ Co- or contravariant functors $R \to Ab$ are exactly the left or right R-modules.
When R is a K-algebra (K a field) we use functors $R \to K\text{-Vect}$ instead.

- When A is a K-algebra and $1 = e_1 + \dots + e_n$ a decomposition of 1_A into a sum of pairwise orthogonal idempotents, A can be seen as a category with n objects, $e_1, \dots, e_n$ and morphisms $\mathrm{Hom}(e_i, e_j) := e_i A e_j$. A functor $M : A \to K\text{-Vect}$ is a module with a vectorspace decomposition $M = \bigoplus_{i=1}^{n} e_i M$ (or $M = \bigoplus_{i=1}^{n} M e_i$).
- Let R be a ring and $\mathcal{C} := R\text{-Mod}$.

Fix $X \in R\text{-Mod}$. $\mathrm{Hom}_R(X,-)$ sends an object Y to $\mathrm{Hom}_R(X,Y) \in Ab$, and an R-homomorphism $f : Y \to Z$ to $\underbrace{\mathrm{Hom}_R(X,f)}_{= f_*} : \mathrm{Hom}_R(X,Y) \longrightarrow \mathrm{Hom}_R(X,Z)$

$$(g : X \to Y) \longmapsto (f \circ g : X \xrightarrow{g} Y \xrightarrow{f} Z)$$

$f_1 : Y \to Z$, $f_2 : Z \to W$ $\Rightarrow$

$$\mathrm{Hom}_R(X, f_2 \circ f_1) : \mathrm{Hom}_R(X,Y) \longrightarrow \mathrm{Hom}_R(X,W)$$
$$(g : X \to Y) \longmapsto (f_2 \circ f_1) \circ g = f_2 \circ (f_1 \circ g)$$

$\Rightarrow$ $\mathrm{Hom}_R(X,-)$ is a covariant functor, $R\text{-Mod} \longrightarrow Ab$.

Similarly, $\mathrm{Hom}_R(-,X)$ is a contravariant functor: $Y \longmapsto \mathrm{Hom}_R(Y,X)$,

$$(g : Z \to Y) \longmapsto \mathrm{Hom}_R(g, X) = g^* : \mathrm{Hom}_R(Y,X) \longrightarrow \mathrm{Hom}_R(Z,X)$$
$$(h : Y \to X) \longmapsto (h \circ g : Z \xrightarrow{g} Y \xrightarrow{h} X)$$

We have shown:

5.6 Proposition: Let R be a ring (or a K-algebra) and $X \in R$-Mod. Then $\mathrm{Hom}_R(X,-): R\text{-Mod} \to Ab$ (or K-Vect) is a covariant functor, and $\mathrm{Hom}_R(-,X): R\text{-Mod} \to Ab$ (or K-Vect) is a contravariant functor.

Not surprisingly, Ext^1_R also provides us with a covariant functor $\mathrm{Ext}^1_R(Z,-)$ (like Hom: covariant $\Leftrightarrow$ first variable fixed) and a contravariant functor $\mathrm{Ext}^1_R(-,X)$ (like Hom: contravariant $\Leftrightarrow$ second variable fixed):

5.7 Proposition: Let R be a ring (or a K-algebra) and $Z, X \in R$-Mod. Then $\mathrm{Ext}^1_R(Z,-): R\text{-Mod} \to Ab$ (or K-Vect) is a covariant functor, and $\mathrm{Ext}^1_R(-,X): R\text{-Mod} \to Ab$ (or K-Vect) is a contravariant functor.

We know already most of the properties: Each $\mathrm{Ext}^1_R(Z,X)$ is an abelian group (we know two explanations of the group structure). The problem sheet "More exercises on Ext^1" tells how morphisms $\alpha: X \to X'$ and $\beta: Z' \to Z$ act on representatives of equivalence classes of extensions

$$0 \to X \to Y \to Z \to 0.$$

Choosing $\alpha = \lambda 1_X: X \to X$ or $\beta = \lambda 1_Z: Z \to Z$ for $\lambda \in K$ gives the vector space structure, which also follows from Theorem 3.3.

Recall from the problem sheet:

$$\begin{tikzcd}
 & 0 \arrow[r] & X \arrow[r] \arrow[d, equal] & \text{pullback} \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d, "\beta"] & 0 \\
\xi: & 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\xi: & 0 \arrow[r] & X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & X' \arrow[r] & \text{pushout} \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

(and this induces maps on equivalence classes of extensions).

When $\alpha = 1_X$ or $\beta = 1_Z$ then the resulting extension is just ξ itself.
We have to check that the constructions are compatible with taking compositions of morphisms, as required in 5.5.
This is implied by properties of pushout or pullback, respectively:
For instance, given $X \xrightarrow{\alpha_1} X' \xrightarrow{\alpha_2} X''$ we can form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_1"'] & Y \arrow[d] \\ X' \arrow[r, "\lrcorner"] \arrow[d, "\alpha_2"'] & PO \arrow[d] \\ X'' \arrow[r, "\lrcorner"] & PO' \end{tikzcd}$$

or

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_2\circ\alpha_1"'] & Y \arrow[d] \arrow[ddr, bend left, "\psi"] & \\ X'' \arrow[r, "\lrcorner"] \arrow[drr, bend right, "\varphi"'] & \widetilde{PO} \arrow[dr, "\exists!\,\tau"] & \\ & & V \end{tikzcd}$$

We check that PO' has the properties of $\widetilde{PO}$: Given V, $\varphi: X'' \to V$ and $\psi: Y \to V$ satisfying commutativity, there is a unique $\tau: \widetilde{PO} \to V$ making everything commutative.

We have to find $\tau': PO' \to V$ doing the same.
$\psi \circ f = \underbrace{\varphi\circ(\alpha_2\circ\alpha_1)}_{=(\varphi\circ\alpha_2)\circ\alpha_1}$ by assumption. PO is pushout wrt α_1 and f $\Rightarrow$ $\exists!$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_1"'] & Y \arrow[d] \arrow[ddr, bend left, "\psi"] & \\ X' \arrow[r] \arrow[d, "\alpha_2"'] \arrow[drr, bend right, "\varphi\circ\alpha_2"] & PO \arrow[dr, "\exists!\,\sigma"] & \\ X'' \arrow[rr, "\varphi"'] & & V \end{tikzcd}$$

Using the pushout property of PO', the existence and uniqueness of τ' follows.
Similarly for pullbacks.

We will see another explanation for functoriality of Ext^1 later on in the context of projective (or injective) resolutions and Ext^n.

The same arguments also tell us that endomorphisms $\alpha: X \to X$ and $\beta: Z \to Z$ act on $Ext^1_R(Z, X)$ turning it into a module over each of the endomorphism rings, of X and of Z. In this way, we get functors from R-modules to $End_R(X)$- or $End_R(Z)$-modules, respectively, as for $Hom_R(Z, X)$.

Therefore, $\mathrm{Hom}_R(X,-)$, $\mathrm{Hom}_R(-,X)$, $\mathrm{Ext}^1_R(X,-)$ and $\mathrm{Ext}^1_R(-,X)$ can be used to compare module categories.

Functors $F: \mathcal{C} \to \mathcal{D}$ are morphisms between categories. We can define F to be an isomorphism, and $\mathcal{C}$ and $\mathcal{D}$ to be isomorphic through F, if and only if there exists a functor $G: \mathcal{D} \to \mathcal{C}$ such that $G \circ F = id_{\mathcal{C}}$ and $F \circ G = id_{\mathcal{D}}$ and both F and G are covariant.
This makes sense, but it restricts the number of categories to be compared in this way too much. We choose a more general definition, equivalence of isomorphism, which is much more interesting and useful.

<u>5.8 Definition:</u> Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G: \mathcal{C} \to \mathcal{D}$ covariant functors. A <u>natural transformation</u> $\eta: F \to G$ is a class of morphisms in $\mathcal{D}$, $\{\eta_X: F(X) \to G(X)\}_{X \in Ob(\mathcal{C})}$, such that for all morphisms $f: X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowright"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

is commutative, that is,

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

A natural transformation η is called <u>a natural isomorphism</u> (or a functorial isomorphism) $:\Leftrightarrow$ $\forall X \in Ob\,\mathcal{C}: \eta_X$ is an isomorphism.
A covariant functor $F: \mathcal{C} \to \mathcal{D}$ is an <u>equivalence of categories</u> if there exists a covariant functor $H: \mathcal{D} \to \mathcal{C}$ and there exist natural isomorphisms

$$\eta: 1_{\mathcal{C}} \to H \circ F \quad \text{and} \quad \theta: 1_{\mathcal{D}} \to F \circ H$$

H then is called <u>quasi-inverse</u> to F and $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent categories.

(When F and H are contravariant functor related by a natural isomorphism η, one calls them <u>dualities</u> instead of equivalences.)

Definition 5.8 is quite a mouthful. Let us try to understand its meaning.
Let Q be a quiver. We have seen that a covariant functor $M: \overset{=\mathcal{C}}{Q} \to \overset{=\mathcal{D}}{k\text{-Vect}}$ is nothing but a representation of the quiver Q over the field k, sending a vertex i to a vectorspace $M(i)$ and an arrow $\alpha: i \to j$ to a k-linear map $M(\alpha): M(i) \to M(j)$. Let N be another functor = representation.
Let $\eta: M \to N$ be a natural transformation from M to N.
This means: For an object = vertex $i \in Q_0$ there is a morphism $\eta_i: M(i) \to N(i)$ such that for all morphisms = paths $p: i \to j$ in Q, the diagram

$$\begin{tikzcd} M(i) \arrow[r, "\eta_i"] \arrow[d, "M(p)"'] \arrow[dr, phantom, "\circlearrowright"] & N(i) \arrow[d, "N(p)"] \\ M(j) \arrow[r, "\eta_j"] & N(j) \end{tikzcd}$$

commutes. Choosing $p = e_i$ (for $i = j$) or $p = \alpha: i \to j$ an arrow we see that this is exactly the condition for η to form a homomorphism between the representations M and N.
This works as well when we view an algebra A as a category and a functor $M: A \to k\text{-Vect}$ as an A-module. Then natural transformations are the same as module homomorphisms.
Generally, one can view functors $\mathcal{C} \to Ab$ or $\mathcal{C} \to k\text{-Vect}$ as modules over the category $\mathcal{C}$.

Next, let us try to understand what $F: \mathcal{C} \to \mathcal{D}$ being an equivalence means.
An isomorphism between vectorspaces or modules is a homomorphism that is both injective and surjective. A functor F qualifying as an isomorphism should be injective and surjective as well – both on the class of objects and on morphism sets. The conditions on an equivalence are a bit more general.

5.9 Definition: Let $F: \mathcal{C} \to \mathcal{D}$ be a covariant functor.

- F is essentially surjective (or dense) $:\Leftrightarrow \forall Y \in Ob(\mathcal{D})\ \exists X \in Ob(\mathcal{C}): Y \simeq F(X)$.
- F is faithful $:\Leftrightarrow \forall X, Y \in Ob(\mathcal{C}): Hom_{\mathcal{C}}(X,Y) \xrightarrow{F} Hom_{\mathcal{D}}(FX, FY)$ injective
- F is full $:\Leftrightarrow \forall X, Y \in Ob(\mathcal{C}): Hom_{\mathcal{C}}(X,Y) \xrightarrow{F} Hom_{\mathcal{D}}(FX, FY)$ surjective
- F is fully faithful $:\Leftrightarrow$ F is full and faithful

These are exactly the properties of an equivalence:

5.10 Theorem: Let $F: \mathcal{C} \to \mathcal{D}$ be a covariant functor. Then F is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Essentially surjective is the property that makes equivalences much more general than isomorphisms of categories.

Example: Let K be a field and $\mathcal{C}$ the category with objects K^n, $n \in \mathbb{N}_0$, and K-linear maps as morphisms. Let $\mathcal{D} := K$-vect, the category of finite-dimensional vector spaces. $Ob(\mathcal{C})$ is a countable set, $Ob(\mathcal{D})$ is not a set, but a proper class. $\mathcal{C}$ is a full subcategory of $\mathcal{D}$. Let $F: \mathcal{C} \to \mathcal{D}$ be the inclusion. F is fully faithful and in linear algebra we have learnt that F is essentially surjective. Hence $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories. And the term "equivalent" really makes sense. In linear algebra we have seen that problems about $\mathcal{D}$ always can be solved in $\mathcal{C}$.

Proof of 5.10:

"$\Rightarrow$" Suppose G is quasi-inverse to F and $\eta: 1_{\mathcal{C}} \to G \circ F$ and $\theta: 1_{\mathcal{D}} \to F \circ G$ are natural isomorphisms.

F is dense: Let $Y \in Ob(\mathcal{D})$. Then $\theta: 1_{\mathcal{D}}(Y) = Y \xrightarrow{\sim} (F \circ G)(Y) = F(G(Y))$.

F is faithful: Let $f \in \mathcal{C}(X_1, X_2)$ be a morphism. By definition of natural transformation, there is a commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r, "\eta_{X_1}", "\sim"'] \arrow[d, "f"'] & (G \circ F)(X_1) \arrow[d, "(G \circ F)(f)"] \\ X_2 \arrow[r, "\sim", "\eta_{X_2}"'] & (G \circ F)(X_2) \end{tikzcd}$$

$\Rightarrow f = \eta_{X_2}^{-1} \circ G(F(f)) \circ \eta_{X_1}$

$\Rightarrow (f \mapsto F(f))$ is an injective map

F is full: Let $g \in \mathcal{D}(Y_1, Y_2)$ be a morphism. Then the commutative diagram

$$\begin{tikzcd} Y_1 \arrow[r, "\theta_{Y_1}", "\sim"'] \arrow[d, "g"'] & (F \circ G)(Y_1) \arrow[d, "(F \circ G)(g)"] \\ Y_2 \arrow[r, "\theta_{Y_2}", "\sim"'] & (F \circ G)(Y_2) \end{tikzcd}$$

shows $\theta_{Y_2} \circ g \circ \theta_{Y_1}^{-1} = (F \circ G)(g) = F(G(g))$. This implies $h \mapsto F(h)$ is surjective, since $\mathcal{D}(Y_1, Y_2) \cong \mathcal{D}(F(G(Y_1)), F(G(Y_2)))$ by $g \mapsto \theta_{Y_2} \circ g \circ \theta_{Y_1}^{-1}$

This shows that F is dense and fully faithful.

"$\Leftarrow$" The converse is more complicated, since we have to find a quasi-inverse for F, which now is assumed to be fully faithful and dense.

Construction of $G: \mathcal{D} \to \mathcal{C}$:

On objects: Let $Y \in Ob(\mathcal{D})$. F is dense $\Rightarrow \exists X \in Ob\,\mathcal{C}: Y \cong F(X)$.

Set $G(Y) := X$. X is not unique, we just choose an X. Fix an isomorphism $\varphi_Y: Y \to F(X)$.

On morphisms: Let $g: Y_1 \to Y_2$ be a morphism in $\mathcal{D}$, $G(Y_1) = X_1$ and $G(Y_2) = X_2$.

F is fully faithful $\Rightarrow \exists$ isomorphism $\Psi_F: \mathcal{C}(X_1, X_2) \xrightarrow{\sim} \mathcal{D}(F(X_1), F(X_2))$

$f \mapsto F(f)$

Set $G(g) := \Psi_F^{-1}(\varphi_{Y_2} \circ g \circ \varphi_{Y_1}^{-1})$.

$$F(X_1) \xrightarrow[\varphi_{Y_1}^{-1}]{\sim} Y_1 \xrightarrow{g} Y_2 \xrightarrow[\varphi_{Y_2}]{\sim} F(X_2)$$

This defines G and G is a functor: $G(1_Y) = \Psi_{F^{-1}}(\varphi_Y \circ 1_Y \circ \varphi_Y^{-1}) = 1_{G(Y)}$, since F is a functor, and for $Y_1 \xrightarrow{g} Y_2 \xrightarrow{h} Y_3$:

$$G(h \circ g) = \Psi_F^{-1}(\varphi_{Y_3} \circ (h \circ g) \circ \varphi_{Y_1}^{-1}) = \Psi_F^{-1}(\varphi_{Y_3} \circ h \circ \varphi_{Y_2}^{-1} \circ \varphi_{Y_2} \circ g \circ \varphi_{Y_1}^{-1}) =$$

$$= \Psi_F^{-1}(\varphi_{Y_3} \circ h \circ \varphi_{Y_2}^{-1}) \circ \Psi_F^{-1}(\varphi_{Y_2} \circ g \circ \varphi_{Y_1}^{-1}) = G(h) \circ G(g),$$ since F is a functor.

Now we are going to show that G is a quasi-inverse to F:

$\varphi = \{\varphi_Y\}_{Y \in Ob(\mathcal{D})}: 1_{\mathcal{D}} \to F \circ G$ is a natural isomorphism because of the commutative diagram

$$\begin{tikzcd} Y_1 \arrow[r, "\varphi_{Y_1}"] \arrow[d, "g"'] & F(X_1) \arrow[r, equal] \arrow[d, "F(f)"] & F(G(Y_1)) \arrow[d, "F(G(g))"] \\ Y_2 \arrow[r, "\varphi_{Y_2}"'] & F(X_2) \arrow[r, equal] & F(G(Y_2)) \end{tikzcd}$$

(for $G(g) = f$)

Construction of a natural isomorphism $\eta: 1_{\mathcal{C}} \to G \circ F$: Let $X \in Ob\,\mathcal{C}$, then $F(X) \in Ob(\mathcal{D})$ and $\varphi_{F(X)}: F(X) \xrightarrow{\sim} F(G(F(X)))$, an isomorphism in $\mathcal{D}$.

F is fully faithful, $\Psi_F: \mathcal{C}(X, G(F(X))) \xrightarrow{\sim} \mathcal{D}(F(X), F(G(F(X))))$

preimage $\eta_X \xmapsto{\Psi_F} \varphi_{F(X)}$

$\eta_X: X \to G(F(X))$ is the Ψ_F-preimage of $\varphi_{F(X)}$

$\varphi_{F(X)}$ is an isomorphism, F is a functor and Ψ_F bijective $\Rightarrow$

$\eta_X: X \xrightarrow{\sim} G(F(X))$ is an isomorphism, too

$\eta = \{\eta_X\}_{X \in Ob(\mathcal{C})}$ is a natural transformation: Let $f: X_1 \to X_2$ be a morphism in $\mathcal{C}$. Then $F(\eta_{X_2} \circ f) = F(\eta_{X_2}) \circ F(f) = \varphi_{F(X_2)} \circ F(f) =$

$$= \varphi_{F(X_2)} \circ F(G(F(f))) \circ \varphi_{F(X_1)} =$$

φ nat transf $1_{\mathcal{D}} \to F \circ G$

$$= F(G(F(f))) \circ F(\eta_{X_1})$$

$$= F(G(F(f)) \circ \eta_{X_1})$$

F fully faithful $\Rightarrow \eta_{X_2} \circ f = G(F(f)) \circ \eta_{X_1} \Rightarrow$ the following diagram commutes:

$$\begin{tikzcd}
X_1 \arrow[r, "\eta_{X_1}", "\sim"'] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & G(F(X_1)) \arrow[d, "G(F(f))"] \\
X_2 \arrow[r, "\eta_{X_2}", "\sim"'] & G(F(X_2))
\end{tikzcd}$$

which shows that $\eta: 1_{\mathcal{C}} \to G \circ F$ is a natural isomorphism.

Hence G is quasi-inverse to F. $\square$