Exercises on natural isomorphisms

(1) Let R be a ring and $\mathcal{C}$ the category of R-modules, $\mathcal{C} = R\text{-Mod}$.
Let $F = id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and $G = Hom_R({}_R R_R, -) : \mathcal{C} \to \mathcal{C}$, and
$\varphi : M \to Hom_R(R, M)$. Show that F and G are functors and
define φ so that it becomes a natural isomorphism $\varphi : F \to G$.

(2) Let R be a ring, $\mathcal{C} = R\text{-Mod}$ and $\mathcal{D} = \mathbb{Z}\text{-Mod}$, the category
of abelian groups. $\oplus$ denotes the coproduct and $\prod$ the product.
(a) Show that for each set $\{M_\lambda \mid \lambda \in \Lambda\}$ of R-modules, there are
natural isomorphisms $Hom_R(\bigoplus_{\lambda\in\Lambda} M_\lambda, -) \simeq \prod_{\lambda\in\Lambda} Hom_R(M_\lambda, -)$

and $Hom_R(-, \prod_{\lambda\in\Lambda} M_\lambda) \simeq \prod_{\lambda\in\Lambda} Hom_R(-, M_\lambda)$.

(b) Show that a direct sum of projective modules is projective
and that a direct product of injective modules is injective.