A quiver Q is a combinatorial objects with vertex set Q_0 and arrow set Q_1.
From Q we get an algebraic object kQ, the path algebra, which has simple modules $S(i)$, $i \in Q_0$, associated with the vertices. And Q_1 can be recovered from the extension groups $\mathrm{Ext}^1(S(i), S(j))$.
When A is any finite dimensional algebra over a field k, then it has, up to isomorphism, finitely many simple modules $S(1), \dots, S(n)$. (Since any simple module must occur in the composition series of the regular module $_AA$, which is finite and unique up to reordering.) Computing all $\mathrm{Ext}^1(S(i), S(j))$, one can determine a quiver for A. What does this quiver Q tell us and what does it fail to tell us?

- $A = \mathrm{Mat}(n\times n, k)$ has one simple module, S, and $\mathrm{Ext}^1_A(S,S) = 0$. So $Q = \bullet$.
  Q does not tell us what is $n = \dim_k S$. It's the same Q for all n.
- $A = \mathbb{Q}(\sqrt{2})$ as a $\mathbb{Q}$-algebra also has quiver $Q = \bullet$. Q doesn't tell us about field extensions involved in defining A.
- $A = k[x]/\langle x^n\rangle$ $(n \geq 2)$ has one simple module, $S = k = A/\langle x\rangle$, and its projective resolution can be chosen as

$$\begin{tikzcd} k[x]/\langle x^n\rangle \arrow[r] \arrow[dr] & k[x]/\langle x^n\rangle \arrow[r] & k \arrow[r] & 0 \\ & \langle x\rangle/\langle x^n\rangle \arrow[u] & & \end{tikzcd}$$

$\Rightarrow \mathrm{Ext}^1_A(S,S) = k$. For any n.
$Q = \bullet\circlearrowleft$ does not identify n.

So, many algebras can have the same quiver. This should not surprise us. After all, there are only countably many quivers, but uncountably many k-algebras, up to isomorphism, unless k is finite or countable.
To phrase it positively: Many k-algebras must be related to each other by having the same quiver, i.e. the same $\mathrm{Ext}^1_A(S(i), S(j))$. Can we construct such algebras, taking the path algebra kQ as a starting point?
What about quotients of kQ? For instance, $k[x] \simeq k(\bullet\circlearrowleft)$ and the quotient algebras $k[x]/\langle x^n\rangle$ also have quiver $\bullet\circlearrowleft$ – except $n=1$. & $k[x]/\langle x\rangle \simeq k$ has quiver $\bullet$ – the arrow $\circlearrowleft$ got lost, since it is in the ideal being factored out.
So, when passing from kQ to kQ/I (I a two-sided ideal), we should make sure, that no arrows are in I. And of course no vertices either.
Moreover, we want kQ/I to be finite dimensional, even if $\dim kQ = \infty$.

4.7 Definition: Let Q be a quiver and K a field. $(KQ)_{\geq n}$ is the ideal of KQ, which has K-basis all paths of length at least n (for some $n \in \mathbb{N}$).
An ideal I of KQ is called <u>admissible</u> $:\Leftrightarrow \exists\, n \geq 2$ such that

$$(KQ)_{\geq n} \subseteq I \subseteq (KQ)_{\geq 2}$$

When I is an admissible ideal of KQ, the quotient algebra KQ/I is called a <u>bound quiver algebra</u>.
A set of generators of I (as two-sided ideal) is called a set of relations.

Here, ideal always means two-sided ideal.
A bound quiver algebra always is finite dimensional because of $I \supseteq (KQ)_{\geq n}$.
KQ itself need not be finite dimensional. If it is, then it is a bound quiver algebra, for large enough n.
The set of relations always can be chosen to be finite.
For $x \in I$, also $e_i x e_j \in I$. Hence one can require each individual relation (= generator) to be a linear combination of paths starting at the same vertex i and ending at the same vertex j.
I is not required to be homogeneous, relations involving paths of different lengths are allowed. Example: $Q =$

$$\begin{tikzcd} \bullet \arrow[r, "\alpha_1"] \arrow[dr, "\beta_1"'] & \bullet \arrow[r, "\alpha_2"] & \bullet \arrow[dr, "\alpha_3"] & \\ & \bullet \arrow[rr, "\beta_2"'] & & \bullet \end{tikzcd}$$

, $I = \langle \alpha_1\alpha_2\alpha_3 - \beta_1\beta_2 \rangle$, so in KQ/I
$\overline{\alpha_1\alpha_2\alpha_3} = \overline{\beta_1\beta_2}$.
(We often write $\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2$ in KQ/I to avoid overloading notation. But we have to remember that paths represent residue classes.)
Relations involving paths of length 0 (vertices) or of length 1 (arrows) are not allowed, since $I \subseteq (KQ)_{\geq 2}$.
$I \subseteq (KQ)_{\geq 1}$ means: $KQ/(KQ)_{\geq 1} \simeq (KQ/I)\big/(KQ/I)/\overline{(KQ)_{\geq 1}}$
residue classes
$\Rightarrow KQ/I$ contains all vertices,
and similarly all arrows, which are a K-basis of $(KQ)_{\geq 1}/(KQ)_{\geq 2}$
$\Rightarrow Q_0$ and Q_1 "exist" in KQ/I. But we do not know, at this point, if Q in KQ/I is related to extensions between simples.

Let us consider some examples of bound quiver algebras:

$Q = \cdot \circlearrowleft \alpha$, $I_n = \langle \alpha^n \rangle$ for some $n \geq 2$, $KQ/I_n \simeq K[x]/\langle x^n \rangle$ is a bound quiver algebra. $n = 1$ is not allowed. But $K[x]/\langle x \rangle \simeq KQ'$, $Q' = \cdot$

$Q = \underset{1}{\cdot} \xrightarrow{\alpha_1} \underset{2}{\cdot} \xrightarrow{\alpha_2} \underset{3}{\cdot} \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} \underset{n}{\cdot}$, $KQ \simeq \begin{pmatrix} K & \cdots & K \\ 0 & \ddots & \vdots \\ 0 & \cdots & K \end{pmatrix}$

$I_1 = \langle \alpha_1 \alpha_2 \cdots \alpha_{n-1} \rangle$

KQ/I_1 is a bound quiver algebra for $n \geq 3$. $KQ/I_1 \simeq \begin{pmatrix} K & \cdots & K \\ 0 & \ddots & \vdots \\ 0 & \cdots & K \end{pmatrix} \Big/ \begin{pmatrix} 0 & \cdots & 0 & K \\ \vdots & \ddots & & 0 \\ & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}$

$I_2 = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_{n-2}\alpha_{n-1} \rangle$ is a bound quiver algebra for $n \geq 3$. $KQ/I_2 \simeq \begin{pmatrix} K & \cdots & K \\ 0 & \ddots & \vdots \\ 0 & \cdots & K \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 & K & \cdots & K \\ 0 & 0 & 0 & K & \cdots \\ \vdots & & \ddots & \ddots & K \\ & & & & 0 \\ 0 & & \cdots & & 0 \end{pmatrix}$

A K-basis of KQ/I_2 is, for instance,

$e_1, \ldots, e_n, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$; its K-dimension is $n + (n-1) = 2n-1$.

By definition, KQ/I is, as algebra, a quotient of the path algebra KQ.
$\Rightarrow$ Every right KQ/I-module M is a right KQ-module as well and thus a representation of the quiver Q. Conversely, a right KQ-module N is a right KQ/I-module iff $I \cdot N = 0$.

Examples: $K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{} \cdots \xrightarrow{1} K$ is a representation of KQ, $Q = \cdot \to \cdot \to \cdots \to \cdot$, but not of KQ/I_1 or KQ/I_2

$K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{0} K \xrightarrow{1} \cdots \xrightarrow{1} K$ is a representation of KQ and of KQ/I_1, but not of KQ/I_2

$K \xrightarrow{1} K \xrightarrow{0} K \xrightarrow{1} K \xrightarrow{0} K \xrightarrow{1} \cdots \to K$ is a representation of all three of them.

Let $A := KQ/I$ with I admissible: $(KQ)_{\geq n} \subseteq I \subseteq (KQ)_{\geq 2}$. Then A is generated, over K, by the residue classes of the paths in Q of length less than n. Hence, A is a finite dimensional K-algebra — even if KQ has infinite dimension.

Let $\pi: KQ \twoheadrightarrow A$ be the quotient map. Then $\underbrace{\pi((KQ)_{\geq 1})}_{J}$ is a two-sided ideal $J \subset A$, and $J^n = 0 \Rightarrow J$ is nilpotent.

Let S be a simple right A-module. SJ is a submodule of S $\Rightarrow$ $SJ = 0$ or $SJ = S$. If $SJ = S$ then $S = SJ = SJ^2 = \ldots = SJ^n = S \cdot 0$ ↯
$\Rightarrow SJ = 0 \Rightarrow S$ is a simple A/J-module.
$A/J \simeq KQ/(KQ)_{\geq 1} = Ke_1 \oplus Ke_2 \oplus \ldots \oplus Ke_n$ when $Q_0 = \{1, \ldots, n\}$.
Consequence: Up to isomorphism, the simple A-modules are $\underset{S(1)}{\overset{}{S_1}}, \ldots, \underset{S(n)}{S_n}$, which correspond to the one-dimensional Q-representations concentrated at the vertices $1, \ldots, n \in Q_0$.

The idempotents $e_1, e_2, \ldots, e_n$ in A define projective right A-modules $P(1) = e_1 A, \ldots, P(n) = e_n A$. Here, $e_i A = e_i (KQ/_I)$ is K-generated by all paths starting at i, but these need not be linearly independent.

Examples:

- $$Q = \begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\ 2 \arrow[dr, "\beta"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd} \qquad I = \langle \alpha\beta - \gamma\delta \rangle$$

KQ-projectives: $e_1 KQ$ has basis $\overset{e_1}{1}, \alpha, \alpha\beta, \gamma, \gamma\delta$
$e_2 KQ$ has basis $2, \beta$
$e_3 KQ$ has basis $3, \delta$
$e_4 KQ$ has basis 4

$A = KQ/I$, $e_1 A$ has basis $1, \alpha, \alpha\beta = \gamma\delta, \gamma$
$e_2 A$ has basis $2, \beta$
$e_3 A$ has basis $3, \delta$
$e_4 A$ has basis 4

- Q any quiver, $I = (KQ)_{\geq 2}$ contains all paths of lengths ≥ 2.
$A = KQ/_I$ has basis $1, \ldots, n \in Q_0$ and $\alpha, \beta, \gamma, \ldots \in Q_1$, i.e. all vertices and all arrows, and nothing else.

Example:
$$Q = \begin{tikzcd} 1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & 3 \arrow[d, "\alpha_3"] \\ n \arrow[u, "\alpha_n"] & n-1 \arrow[l, "\alpha_{n-1}"] & \vdots \arrow[l] \end{tikzcd}$$

$I = (KQ)_{\geq 2}$, $A = KQ/I$, $P(i) = e_i A$ has basis e_i, α_i, i.e. $\dim P(i) = 2$.
$\dim S(i) = 1 \Rightarrow$ projective resolutions cannot be finite. why?

Some examples with projective resolutions:

$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 / \langle \alpha\beta \rangle$ $\quad P(1): 1, \alpha, \; P(2): 2, \beta, \; P(3): 3$

$0 \to P(3) \to P(2) \to P(1) \to S(1) \to 0$

$S(3)$, $S(2)$

pdim $S(1) = 2$

$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\mu} 4 / \langle \alpha\beta\mu \rangle$ $\quad P(1): \quad P(2): \quad P(3): \quad P(4):$

$0 \to P(4) \to P(2) \to P(1) \to S(1) \to 0 \quad$ pdim $S(1) = 2$

continue the series

$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\mu} 4 / \langle \alpha\beta, \beta\mu \rangle$ $\quad P(1): \quad P(2): \quad P(3): \quad P(4):$

$0 \to P(4) \to P(3) \to P(2) \to P(1) \to S(1) \to 0 \quad$ pdim $S(1) = 3$

Continue the series

Why are these projective resolutions of minimal length?

Proposition 4.4 and Corollary 4.5 do not carry over to bound quiver algebras. But theorem 4.6 does so, basically with the same proof. Check!

<u>4.8 Corollary:</u> Let $A = kQ/I$ be a bound quiver algebra, $a, b \in Q_0$. Then the k-dimension of $\mathrm{Ext}^1_A(S(a), S(b))$ equals the number of arrows $\alpha \in Q_1$ from a to b.

So, as for path algebras, the combinatorial datum Q determines, and is determined by, the structural datum $\mathrm{Ext}^1_A(S,T)$ of extension groups (as k-spaces) between simple modules S and T.