§4. Examples (path algebras and bound quiver algebras)

In the first part of this chapter we consider representations of (path algebras of) quivers. Our main interest is in finite dimensional path algebras and finite dimensional modules.

Recall the definitions of quivers and path algebras. (Note that there is no unified notation for multiplication of paths and thus our path algebra may be the opposite of the path algebra elsewhere and representations of quivers may be left or right modules over the path algebra, depending on conventions).

4.1 Definition: A *quiver* is a quadruple $Q=(Q_0, Q_1, s, t)$ where Q_0 is a non-empty finite set (whose elements are called *vertices*), Q_1 is a finite set (whose elements are called arrows) and s and t are functions $s,t: Q_1 \longrightarrow Q_0$ (sending an arrow α to its *source* $s(\alpha)$ or its *target* $t(\alpha)$).

or starting vertex — or terminal vertex

Examples: $\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ}$, $s(\alpha)=1$, $t(\alpha)=2$

$\underset{1}{\circ} \circlearrowleft^{\alpha}$, $s(\alpha)=t(\alpha)=1$

$\overset{1}{\circ} \quad \overset{2}{\circ}$, $Q_1=\emptyset$

When $t(\alpha)=s(\beta)$ as in $\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ} \xrightarrow{\beta} 3$ then α and β can be concatenated (multiplied) to form the path $\alpha\beta$ which starts in $1=s(\alpha\beta)$ and ends in $2=t(\alpha\beta)$. $\alpha\beta$ is a path of length two, elements in Q_1 are paths of length 1, and vertices are considered to be path of lengths 0 (lazy paths). We write e_1 for the lazy path at vertex 1, and so on.

Longer paths p and q with $t(p)=s(q)$ can be multiplied to form the path pq, and so on. But we cannot multiply p with q when $t(p) \neq s(q)$. We can, however, linearise the situation, as one does when defining a group algebra:

Let K be any field. The <u>path algebra</u> KQ is a K-vector ~~basis~~ space with basis the paths in Q (including the trivial paths e_i). Multiplication is the linear extension of the multiplication of paths

$$p \cdot q := \begin{cases} pq & (\text{path}) \text{ if } t(p) = s(q) \\ 0 & \text{if } t(p) \neq s(q) \end{cases}$$

KQ is a K-algebra. Its unit element is $1 = \sum_{i \in Q_0} e_i$

Examples: $K(1 \xrightarrow{\alpha} 2) \simeq \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ (upper triangular matrices)

$K(\bullet) \simeq K$

$K(\bullet \circlearrowleft) \simeq K[x]$ (polynomial ring)

$K(\bullet \ \bullet) \simeq K \oplus K$

$K\left(\circlearrowleft^{\alpha}_{\bullet} \circlearrowleft_{\beta}\right) \simeq K\langle x, y \rangle$ (free noncommutative algebra)

<u>4.2 Definition:</u> A <u>representation</u> of the quiver Q over the field K consists of two maps $V: Q_0 \to K\text{-vect}$, $i \mapsto V(i)$ a (fin dim) K-vector space

and $V: Q_1 \to K\text{-vect}$, $(\alpha: i \to j) \mapsto V(\alpha): V(i) \to V(j)$ K-linear

Examples: $K \xrightarrow{1} K$, $K \xrightarrow{0} 0$, $K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} K^2$,

$K \circlearrowleft (\lambda)$, K, $\overset{K}{\bullet}$ $\overset{K^2}{\bullet}$

Each vertex gets a vector space and an arrow α gets a linear map $V(\alpha): V(s(\alpha)) \to V(t(\alpha))$.

Representations of Q over K correspond to right KQ-modules, ie left $(KQ)^{op} \simeq K(Q^{op})$-modules:

Given V a right module. Set $V(i) := Ve_i$ and $V(\alpha): V(i) \to V(j)$, for $i \xrightarrow{\alpha} j$, the linear map induced by right multiplication with α.

Conversely, given a representation $V = (V(i), V(\alpha))$, set $\tilde{V} := \bigoplus_{i \in Q_0} V(i)$ and equip $\tilde{V}$ with a KQ-right module structure by letting α act as $V(\alpha)$ and extending this to all paths and linear combinations thereof.

4.3 Definition: Let U and V be representations of Q over k. A homomorphism $\varphi: U \to V$ is a tuple of linear maps $\varphi(i): U(i) \to V(i)$ such that for each arrow $\alpha: i \to j$ the following diagram commutes

$$\begin{tikzcd} U(i) \arrow[r, "U(\alpha)"] \arrow[d, "\varphi(i)"'] \arrow[dr, phantom, "\circlearrowleft"] & U(j) \arrow[d, "\varphi(j)"] \\ V(i) \arrow[r, "V(\alpha)"] & V(j) \end{tikzcd}$$

φ is an isomorphism $\Leftrightarrow$ all $\varphi(i)$ are isomorphisms

φ is an inclusion and U is a subrepresentation of V $\Leftrightarrow$ all $\varphi(i)$ are inclusions

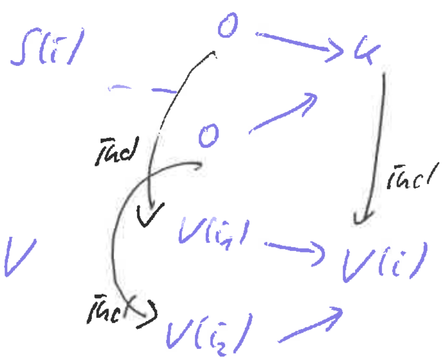
Examples of representations:

For $i \in Q_0$, let $S(i)$ be the representation V with $V(i) = k$ and $V(j) = 0$ for $j \neq i$. Then $S(i)$ is a simple representation: For $U \subset S(i)$, all $U(j)$ must be zero except $U(i)$, which may be k, but then $U = S(i)$.

When kQ is finite dimensional, then $S(1), \dots, S(n)$ is a complete set of simple representations: In fact, $\dim kQ < \infty$ means there are no oriented paths in Q. Hence, there are sinks (where no arrow starts) and sources (where no arrow ends) in Q_0.

When $V(i) \neq 0$ and i is a sink, then $S(i) \subset V$, a subrepresentation:

Q looks like



(When $V(i) = 0$ for all sinks i, one can modify this argument, using the support of V, i.e. the subquiver with Q_0' the set of vertices i with $V(i) \neq 0$.)

When kQ has infinite dimension, then there can be many simple representations, as the example $k(\cdot\circlearrowleft)$ shows.

How do projective representations look like?

In KQ, $1 = e_1 + \dots + e_n$, a sum of pairwise orthogonal idempotents, when $Q_0 = \{1, \dots, n\}$. This can be used to decompose A_A as follows:

$A_A = e_1 A \oplus e_2 A \oplus \dots \oplus e_n A$ (as right module). $\qquad \underset{\shortparallel}{A_A} \; (KQ)_{KQ}$

Here, $e_i A = \{$ linear combinations of paths $e_i p$, p any path$\}$

$\qquad = \{$ linear combinations of paths starting at vertex $i\}$

When KQ is finite dimensional, we get for $P(i) := e_i A$:

$$\dim_K P(i) = \# \text{paths starting at vertex } i$$

and we know a K-basis of $P(i)$.

The support of $P(i)$ then contains all vertices j such that there is a path from i to j. i is the source of all paths contributing to the basis of $P(i)$.

Examples:
$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[dr, "\gamma"'] & 3 \\ & & 4 \end{tikzcd}$$

$P(1)$ has basis $e_1, \alpha, \alpha\beta, \alpha\gamma$

$P(2)$ has basis e_2, β, γ

$P(3)$ has basis e_3, $P(4)$ has basis e_4

$P(3)$ and $P(4)$ are simple, and there are injective homomorphisms

$$\begin{tikzcd} P(3) \arrow[dr, hook, "\beta\cdot"] & & \\ & P(2) \arrow[r, hook, "\alpha\cdot"] & P(1) \\ P(4) \arrow[ur, hook, "\gamma\cdot"'] & & \end{tikzcd}$$

given by left multiplication with β, γ, α respectively:

eg $\qquad P(3) \longrightarrow P(2)$

$\qquad e_3 \longmapsto \beta e_3 \in e_2 A$

Interesting observation: When $\alpha : i \to j$ is an arrow in Q_1, then left multiplication by $\alpha \in KQ$ defines an injective homomorphism of right modules $e_j A \longrightarrow e_i A$

$$e_j p = p \longmapsto \alpha p = e_i \alpha e_j p$$

This helps us to compute projective resolutions of the simple KQ-representations $S_1, \dots, S_n$:

Let $i \in Q_0$. We have to distinguish two cases:

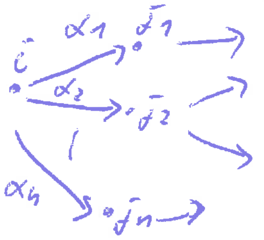
First case: i is a sink in Q, the only path starting at i is e_i.

$\Rightarrow$ $P(i) = S(i)$, this representation is simple and projective

$\Rightarrow$ $\Omega_1(S(i)) = 0$ and as a projective resolution we can choose

$$0 \to 0 \to P(i) \xrightarrow{id} S(i) \to 0$$

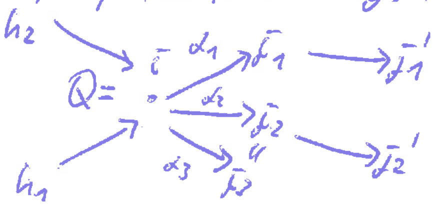
Second case: i is not a sink in Q, there are paths of length ≥ 1 starting at i.



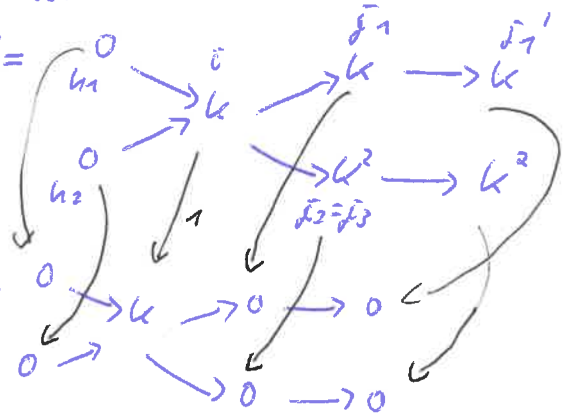
Let $\alpha_1, \dots, \alpha_n$ be the arrows starting at i, and $j_1, \dots, j_n$ be the endpoints of $\alpha_1, \dots, \alpha_n$, respectively (Note that these need not be pairwise different: There can be double, or multiple, arrows: $i \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} j_1 = j_2$)

How does $P(i)$ look like? It has a k-basis consisting of all paths starting at i. Thus at vertex j, the vector space is $k^{n(j)}$, where $n(j)$ counts the paths from i to j. For instance:

$Q =$



$P(i) =$



and $S(i) =$

There is a surjective homomorphism $P(i) \xrightarrow{\varphi} S(i)$, which is $id: k \to k$ at vertex i, and 0 elsewhere. Check that all diagrams commute.

$\Rightarrow \exists$ ses: $0 \to \Omega_1(S(i)) \to P(i) \xrightarrow{\varphi} S(i) \to 0$ and we have to determine $\Omega_1(S(i))$.

Claim: $\Omega_1(S(i)) \cong \bigoplus_{h=1}^{n} P(j_h)$. In particular: $\Omega_1(S(i))$ is projective.

Proof of claim: $\Omega_1(S(i))$ is a subrepresentation of $P(i)$. By the definition of φ, it has a basis consisting of all paths p starting at i and having length ≥ 1. Such a path p must be a product $\alpha_1 p_1$ or $\alpha_2 p_2$ or $\dots$ or $\alpha_n p_n$, i.e. p factors through exactly one of the arrows $\alpha_1, \alpha_2, \dots, \alpha_n$, respectively.

Recall $P(i) = e_i A$ (for $A = kQ$).

Then $\Omega_1(S(i)) = \alpha_1 A \oplus \alpha_2 A \oplus - \oplus \alpha(u) A$: In fact, each basis element p is in exactly one module $\alpha_h A$, and then p factors through α_h, and no path can factor through α_h and another $\alpha_{h'}$.

Now recall that α_h defines an injective homomorphism

$$\begin{aligned} P(t(\alpha_h)) = e_{t(\alpha_h)} A &\longrightarrow e_{s(\alpha_h)} A = e_i A \quad \text{and this has image } \alpha_h A. \\ \text{path } q &\longmapsto \alpha_h \cdot q \end{aligned}$$

This proves the claim.

<u>4.4 Proposition:</u> Let $A = kQ$ be the path algebra of a quiver Q and $S(i)$ a simple kQ-representation, concentrated at vertex $i \in Q_0$.

(a) When i is a sink, then $S(i) = P(i)$ is simple and projective, $\operatorname{pdim} S(i) = 0$.

(b) When i is not a sink, then $\operatorname{pdim} S(i) = 1$, and $\Omega_1(S(i))$ is a direct sum of projective modules $P(j)$, where j runs through the endpoints $t(\alpha)$ for $\alpha \in Q_1$ with $s(\alpha) = i$.

We can extend 4.4 to more general modules using a technique we have seen in the proof of 3.10: When $0 \to X \to Y \to Z \to 0$ is a ses and we have resolved X and Z then we can form a commutative diagram

$$(*)\quad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & K_X \arrow[r] \arrow[d] & K_Y \arrow[r] \arrow[d] & K_Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P_X \arrow[r] \arrow[d] & P_Y \arrow[r] \arrow[d] & P_Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

(with $P_Y = P_X \oplus P_Z$)

We apply that to the situation when X and Z are simples as in 4.4.

Then $K_X = \Omega_1(X)$ and $K_Z = \Omega_1(Z)$ are projective (perhaps even 0).

Since $(*)$ is exact, $K_Y \cong K_Z \oplus K_X$ is projective, too.

$\Rightarrow \operatorname{pdim} Y \leq 1$.

More generally: $\operatorname{pdim} X \leq 1$ and $\operatorname{pdim} Z \leq 1 \Rightarrow \operatorname{pdim} Y \leq 1$.

4.5 Corollary: Let M be finite dimensional kQ-module with a composition series only containing simple composition factors $S(1), -, S(n)$ as in 4.4. Then $\operatorname{pdim} M \leq 1$.
(When $\dim kQ < \infty$, all finite dimensional modules are of this form.)

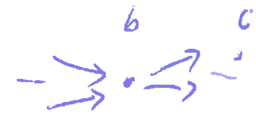
We also can compute extensions between simple modules: Let $S(a)$ and $S(b)$ be simple modules, concentrated at vertices a and b ($a = b$ is allowed. Then $\exists$ $0 \to \bigoplus_{\substack{\alpha_1, -, \alpha_n \in Q_1 \\ s(\alpha_1) = - = s(\alpha_n) \\ = a}} P(t(\alpha_i)) \xrightarrow{f} P(a) \xrightarrow{g} S(a) \to 0$

Apply $\operatorname{Hom}_{kQ}(-, S(b))$ to get the exact sequence as in 3.10:

$$0 \to \operatorname{Hom}_{kQ}(S(a), S(b)) \xrightarrow{g^*} \operatorname{Hom}_{kQ}(P(a), S(b)) \xrightarrow{f^*} \operatorname{Hom}_{kQ}(\oplus P(t(\alpha_i)), S(b)) \to$$
$$\xrightarrow{\delta} \operatorname{Ext}^1_{kQ}(S(a), S(b)) \xrightarrow{g^*} \underbrace{\operatorname{Ext}^1_{kQ}(P(a), S(b))}_{=0 \ (P(a) \text{ projective})} \xrightarrow{f^*} \underbrace{\operatorname{Ext}^1_{kQ}(\oplus P(t(\alpha_i)), S(b))}_{=0}$$

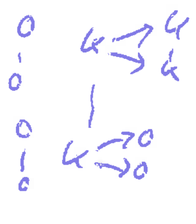
(which agrees with the information provided by Theorem 3.3, since two Ext^1-spaces vanish).

Claim: For $c \in Q_0$, $\operatorname{Hom}_{kQ}(P(c), S(b)) = \begin{cases} k, & c = b \\ 0, & \text{otherwise} \end{cases}$

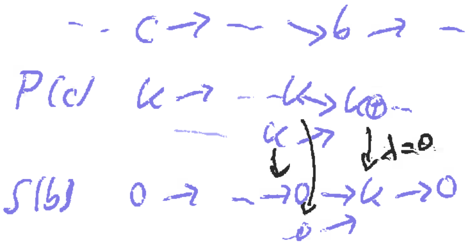
Indeed: $P(c) =$  or 

$S(b) =$ 

When there is no path from c to b, then at b, $P(c)$ has vectorspace 0 and there is only the zero map from 0 to k.



when $c = b$, we know already the unique (up to scalars) map
when $c \neq b$, but there are paths from c to b, then



then only for $d = 0$, the diagram commutes.

This tells us: $\mathrm{Hom}_{kQ}(\bigoplus_i P(t(\alpha_i)), S(b)) = k^d$, where $d = \#\alpha_i : t(\alpha_i) = b$

and also $\mathrm{Hom}_{kQ}(P(a), S(b)) = \begin{cases} k, & a=b \\ 0, & \text{otherwise} \end{cases}$

By Schur's Lemma: $\mathrm{Hom}_{kQ}(S(a), S(b)) = \begin{cases} k, & a=b \\ 0, & a\neq b \end{cases}$
(and direct computation)

Result:

When $a \neq b$: $0 \to 0 \to 0 \to k^d \to \mathrm{Ext}^1_{kQ}(S(a), S(b)) \to 0$

When $a = b$: $0 \to k \to k \to k^d \to \mathrm{Ext}^1_{kQ}(S(a), S(b)) \to 0$

by exactness: this is mono, hence iso

$\Rightarrow$ In both cases: $\mathrm{Ext}^1_{kQ}(S(a), S(b)) = k^d$ (as k-vectorspace), where d is the number of arrows $\alpha \in Q_1$ starting in a and ending in b. When $a=b$ these α are loops.

<u>4.6 Theorem:</u> Let kQ be the path algebra of the quiver over k, $a, b \in Q_0$ and $S(a)$ and $S(b)$ the one-dimensional simple representations associated with the vertices a and b, respectively. Then the k-dimension of $\mathrm{Ext}^1_{kQ}(S(a), S(b))$ equals the number of arrows $\alpha \in Q_1$ from a to b.

This implies several nice or even surprising statements:

- We can read off $\dim_k \mathrm{Ext}^1(S(a), S(b))$ from the quiver.
- This dimension does not depend on the choice of the field k.
- The number of arrows from a to b, in Q_1, or loops at a, in Q_1, is completely determined by the first extensions between one-dimensional simples associated with vertices. In particular, when kQ is finite-dimensional, the extensions between simples determine Q (and kQ). Therefore, in this case, $kQ \neq kQ'$ when Q and Q' are different, or in other words we cannot write kQ as kQ' for a different quiver $Q' \neq Q$.

When $\dim \underset{A}{\underbrace{kQ}} < \infty$, finding the quiver of A is as easy or as complicated as finding the simple modules and the extensions between them.