Theorem 3.3 identifies $\mathrm{Ext}_A^1(M,X)$ with a quotient of $\mathrm{Hom}_A(K,X)$, where $0\to K\to P\to M\to 0$ is an ses with P projective. The proof used several times that P is projective. In fact, if P would not be projective the result would be wrong. We will see shortly how it would go wrong. Independently of that it generally makes sense to choose any ses, apply a covariant or a contravariant Hom and try to understand what happens on the right hand side when surjectivity fails.
This may look like making everything even more complicated, but the answer will turn out to be both easier and more interesting than expected. We will receive crucial support from the snake in the Snake Lemma 1.A.3.

3.9 Theorem: Let M be an A-module and $0\to X\xrightarrow{f} Y\xrightarrow{g} Z\to 0$ an ses of A-modules. Then there is an exact sequence

$$0\to \mathrm{Hom}_A(M,X)\xrightarrow{f_*}\mathrm{Hom}_A(M,Y)\xrightarrow{g_*}\mathrm{Hom}_A(M,Z)\xrightarrow{\delta}\mathrm{Ext}_A^1(M,X)\xrightarrow{f_*}\mathrm{Ext}_A^1(M,Y)\xrightarrow{g_*}\mathrm{Ext}_A^1(M,Z)$$

(f_* and g_* as usual means: compose with f or g respectively on homomorphisms, by 3.3 this should work on Ext as well, but we have to check, and δ comes from 1A.3)

Proof: In order to be able to apply Theorem 3.3, we choose $0\to K\xrightarrow{\alpha} P\xrightarrow{\beta} M\to 0$ with P projective $\rightsquigarrow$ $0\to\mathrm{Hom}_A(M,X)\to\mathrm{Hom}_A(P,X)\to\mathrm{Hom}_A(K,X)\to\mathrm{Ext}_A^1(M,X)\to 0$ and similarly for Y and Z, respectively, by 3.3. We put this into a large diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_A(M,X) \arrow[r, "f_*"] \arrow[d, "\beta^*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(M,Y) \arrow[r, "g_*"] \arrow[d, "\beta^*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(M,Z) \arrow[d, "\beta^*"] \arrow[dddll, bend left, "\delta"] & \\
0 \arrow[r] & \mathrm{Hom}_A(P,X) \arrow[r, "f_*"] \arrow[d, "\alpha^*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(P,Y) \arrow[r, "g_*"] \arrow[d, "\alpha^*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(P,Z) \arrow[r] \arrow[d, "\alpha^*"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(K,X) \arrow[r, "f_*"] \arrow[d] & \mathrm{Hom}_A(K,Y) \arrow[r, "g_*"] \arrow[d] & \mathrm{Hom}_A(K,Z) \arrow[d] & \\
 & \mathrm{Ext}_A^1(M,X) \arrow[r, "f_*"] \arrow[d] & \mathrm{Ext}_A^1(M,Y) \arrow[r, "g_*"] \arrow[d] & \mathrm{Ext}_A^1(M,Z) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here, the vertical exact sequences are given by 3.3., the maps α^* and β^* are precomposing with α or β, respectively. f_* and g_* in the upper three rows are post-composing with f or g, respectively. This commutes with precomposing
$\Rightarrow$ the upper squares are commutative diagrams
$\Rightarrow$ there are induced maps on the cokernels, the Ext^1-groups $\Rightarrow$ in the bottom row we may write f_* and g_* for the induced maps, and we get commutative squares there as well.
The second row of Hom's is exact, since P is projective and 3.2 applies.
The top row consists of kernels of the three α^* maps and the bottom row of cokernels $\Rightarrow$ 1A.3 applies and the connecting homomorphism δ exists and produces the exact sequence claimed in 3.9. $\square$

We can check explicitly what δ: $Hom_A(M,Z) \to Ext^1_A(M,X)$ is doing.
Let $\varphi: M \to Z$ be a homomorphism and form the diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "\tau"'] & P \arrow[r, "\beta"] \arrow[d, "\psi"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

where ψ exists since P is projective, and τ is the restriction of ψ to K.

Recall the construction of δ in the proof of 1A.3:

$\varphi: M \to Z$

$\downarrow$

choose lift $\longleftarrow$ $\overset{\beta^*}{\mapsto}$ $\varphi\circ\beta: P \to M \to Z$
$\psi: P \to Y$
such that $g\circ\psi = \varphi\circ\beta$

$\downarrow$

$f\circ\tau = \psi\circ\alpha: K \to P \to Y$
$K \to X \to Y$
$\tau: K \to X$

$\downarrow$

$\bar{\tau}$ residue class in $Ext^1_A(M,X)$

so: $\delta(\varphi) = \bar{\tau}$

This looks very natural.

To get the corresponding extension we can use the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "\tau"'] & P \arrow[r, "\beta"] \arrow[d, "\pi"'] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r] & 0
\end{tikzcd}$$

where E is the pullback of

$$\begin{tikzcd}
 & M \arrow[d, "\varphi"'] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

and π exists since E is the pullback.

By 3.7,

$$\begin{tikzcd}
K \arrow[r, "\alpha"] \arrow[d, "\tau"] & P \arrow[d] \\
X \arrow[r] & E
\end{tikzcd}$$

is a pushout square.

$\Rightarrow$ The middle row is the extension of M by X corresponding to τ in Theorem 3.3.

Since $Ext^1_\Lambda(M,Y)$ may be non-zero, $Ext^1_\Lambda(M,X)$ need not be the cokernel of g_* in the situation of 3.9.
In that respect, 3.3 gives more precise information. On the other hand, 3.9 gives a nice connection between different Ext-groups, building on the given ses of modules in the same way as the sequence of Hom-groups.
And it invites speculations. Without knowing the definition or meaning of Ext^2, Ext^3, and so on, can you formulate a theorem about all of these, extending 3.9?
We will come back to that, of course.
We could have proven a "dual" version of 3.3, using $0 \to M \to I \to C \to 0$ with I injective. But this will be covered by later results so we omit it here.
We are however, going now to state and prove a "dual" version of 3.9, with M in the second variable, i.e. using the contravariant Hom. This could be derived from the "dual" 3.3, proceeding as in the proof of 3.9, always using injectives instead of projectives. Instead we will use different arguments that are both interesting and useful in themselves.

3.10 Theorem: Let N be an A-module and $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ an ses of A-modules. Then there is an exact sequence

$$0 \to \mathrm{Hom}_A(Z,N) \xrightarrow{g^*} \mathrm{Hom}_A(Y,N) \xrightarrow{f^*} \mathrm{Hom}_A(X,N) \xrightarrow{\delta} \mathrm{Ext}^1(Z,N) \xrightarrow{g^*} \mathrm{Ext}^1_A(Y,N) \xrightarrow{f^*} \mathrm{Ext}^1_A(X,N)$$

In the situation of Theorem 3.3, the ses would be $0 \to K \to P \to M \to 0$ with P projective. Then $\mathrm{Ext}^1_A(P,N)=0$ and $\delta : \mathrm{Hom}_A(K,N) \to \mathrm{Ext}^1(M,N) = \mathrm{Coker}\, f^*$ is surjective. For general Y, $\mathrm{Ext}^1(Y,N)$ may be $\neq 0$ and δ in general may not be surjective. More precisely, δ surjective $\Leftrightarrow g^* = 0 : \mathrm{Ext}^1_A(Y,N) \to \mathrm{Ext}^1_A(X,N)$.

Proof of 3.10: We will again rely on the snake's help, but we have to find the right situation. If we would be interested only in $\mathrm{Ext}^1_A(Z,N)$ we could $0 \to K_Z \to P_Z \to Z \to 0$ with P_Z projective. Now we should do the same for X and for Y, but in such a way that we can relate these data to each other. This is possible: Choose first $P_X \xrightarrow{\varepsilon_X} X \to 0$ and $P_Z \xrightarrow{\varepsilon_Z} Z \to 0$, P_X, P_Z projective.

Claim: There is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & K_X \arrow[r] \arrow[d] & K_Y \arrow[r] \arrow[d] & K_Z \arrow[d] & \\
0 \arrow[r] & P_X \arrow[r] \arrow[d, "\varepsilon_X"'] & P_Y \arrow[r] \arrow[d, "\varepsilon_Y"] & P_Z \arrow[r] \arrow[d, "\varepsilon_Z"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"'] \arrow[d] & Y \arrow[r, "g"'] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $P_X \to X$ and $P_Z \to Z$ are the maps already chosen.

Proof of claim: Let $P_Y := P_X \oplus P_Z$, $P_X \to P_Y$ the inclusion and $P_Y \to P_Z$ the projection. We have to define $\varepsilon_Y : P_Y \to Y$. $P_Y \xrightarrow{proj} P_Z \xrightarrow{\varepsilon_Z} Z$ lifts along $Y \to Z$ to $P_Y \xrightarrow{\varepsilon_1} Y$ where $P_X \to 0$ and $P_Z \xrightarrow{\varepsilon} Y$ lifts ε_Z. The summand P_X of P_Y is sent to 0, by ε_1, but we can map $P_X \to X$ by ~~ε_X~~ $f \circ \varepsilon_X$.

Let $P_Y = P_X \oplus P_Z \xrightarrow{\varepsilon_Y} Y$ be given by $(f\circ\varepsilon_X, \varepsilon)$. The squares involving ε_Y are commutative by definition of ε_Y. We have to check that ε_Y is surjective:

Let $y_0 \in Y$. Then $g(y_0) \in Z$ has an ε_Z-preimage $z_0 \in P_Z$, and $g(\varepsilon_Y(0,z_0)) = g(y_0)$.
$\Rightarrow \exists x_0 \in X : g(\varepsilon_Y(0,z_0)) - g(y_0) = f(x_0)$. Let $x_0' \in P_X$ be an ε_X-preimage of x_0. Then $\varepsilon_Y(x_0', z_0) = y_0$.
Set $K_Y := \mathrm{Ker}(\varepsilon_Y)$. A diagram chase completes the proof of the claim. ✓
Now we apply $\mathrm{Hom}_A(-,N)$ to the diagram in the claim and get:

$$\begin{tikzcd}
 & \mathrm{Hom}_A(Z,N) \arrow[r, "g^*"] \arrow[d] & \mathrm{Hom}_A(Y,N) \arrow[r, "f^*"] \arrow[d] & \mathrm{Hom}_A(X,N) \arrow[d] \arrow[ddll, bend left, "\delta"] & \\
0 \arrow[r] & \mathrm{Hom}_A(P_Z,N) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_Y,N) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_X,N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(K_Z,N) \arrow[r] \arrow[d] & \mathrm{Hom}_A(K_Y,N) \arrow[r] \arrow[d] & \mathrm{Hom}_A(K_X,N) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Ext}^1_A(Z,N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(Y,N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(X,N) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The exact columns are provided by Theorem 3.3. Exactness of the second and the third row is left exactness of Hom, taking into account that the second row in the claim is split exact. Everything else is done by the Snake Lemma. □

A crucial tool in our proofs has been to form sequences like $0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0$ with P projective. It will make a lot of sense to iterate this construction: K also is a quotient of a projective module, say $P' \xrightarrow{g'} K \to 0$, and the kernel K' is a ~~proj~~ quotient of a projective module P'', and so on. $K = \mathrm{Ker}(g)$ and $K = \mathrm{Im}(g')$ implies

$$\begin{tikzcd}
P' \arrow[rr, "f\circ g'"] \arrow[dr, "g'"'] & & P \arrow[r, "g"] & M \arrow[r] & 0 \\
 & K = \mathrm{Im}(g') \arrow[ur, "f"'] & & &
\end{tikzcd}$$

is exact, with P and P' projective

3.11 Definition: Let M be an A-module. A projective resolution of M is a (finite or infinite) exact sequence $0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ (for $n \in \mathbb{N}_0$) or $\cdots \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ where all terms P_i are projective.

The kernel of f_i, which equals the image of f_{i+1}, is denoted by $\Omega_{i+1}(M)$ and called $i+1$-st syzygy of M. ($\Omega_0(M) = M$)

An injective resolution of M is a (finite or infinite) exact sequence $0 \to M \xrightarrow{g_0} I_0 \xrightarrow{g_1} I_1 \xrightarrow{g_2} \cdots \to I_m \to 0$ (for $m \in \mathbb{N}_0$) or $0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to \cdots$ where all terms I_j are injective.

The cokernel of ~~g_j~~ g_j, which equals the kernel of g_{j+1}, is denoted by $\Omega^{j+1}(M)$ and called $j+1$-st cosyzygy of M. Notation: $\Omega^0(M) = M$.

(Since 0 is both projective and injective, some P_i or I_j can be zero.
Since the injective resolution goes in the opposite direction, one sometimes calls it a coresolution.)

Each module M has a projective resolution and an injective resolution:
Given M, choose $P_0 \xrightarrow{f_0} M$ surjective, with P_0 projective.
$\rightsquigarrow$ $0 \to \Omega_1(M) \to P_0 \xrightarrow{f_0} M \to 0$ is exact. Choose $P_1 \to \Omega_1(M)$ surjective and get $0 \to \Omega_2(M) \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ exact.
When $\Omega_n(M)$ happens to be projective we set $P_n := \Omega_n(M)$ and we have found a finite projective resolution.
Already P_0 is far from being unique, and M may have infinite projective resolutions and finite ones of different lengths.
When M is projective we can choose $P_0 = M$, $f_0 = id_M$, and the resolution $0 \to M \xrightarrow{id} M \to 0$ is the shortest possible projective resolution.
Conversely, if $0 \to P_0 \xrightarrow{f_0} M \to 0$ is a projective resolution, then f_0 must be an isomorphism and M is projective.
Similarly: M injective $\Leftrightarrow$ $\exists$ injective resolution $0 \to M \xrightarrow{g_0} I_0 \to 0$

General idea: The length of the shortest projective / injective resolution of M should tell us how "close" M is to being projective / injective. What "close" means isn't clear yet, but we should hope that it is related to the vanishing of certain Ext-groups.

3.12 Definition: The projective dimension of M, denoted by $p\dim(M)$ or $\mathrm{proj}\dim(M)$, is $\inf\{n \in \mathbb{N} : \exists \text{ proj resolution } \underbrace{0 \to P_n \to \dots \to P_{n-1} \to P_0 \to M \to 0}_{\text{length } n}\} \in \mathbb{N} \cup \{\infty\}$

The injective dimension of M, denoted by $i\dim(M)$ or $\mathrm{inj}\dim(M)$ is

$\inf\{m \in \mathbb{N} : \exists \text{ inj resolution } 0 \to M \to I_0 \to I_1 \to \dots \to I_m \to 0\} \in \mathbb{N} \cup \{\infty\}$

So: M projective $\Leftrightarrow$ $p\dim(M) = 0$

M injective $\Leftrightarrow$ $i\dim(M) = 0$

$p\dim(M) = \infty \Leftrightarrow M$ has no finite projective resolution

$i\dim(M) = \infty \Leftrightarrow M$ has no finite injective resolution

Here are a few examples:

- $0 \to \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a projective resolution of the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$. Free modules over $\mathbb{Z}$ are torsion free ($n \cdot x = 0$ for $n \neq 0 \Rightarrow x = 0$) $\Rightarrow$ $\mathbb{Z}/n\mathbb{Z}$ is not a direct summand of a free module, hence not projective $\Rightarrow p\dim(\mathbb{Z}/n\mathbb{Z}) = 1$.
- Let $A = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$, $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (as in the exercises)

  $P_1 = Ae_1$ and $P_2 = Ae_2$ are indecomposable projective. P_1 is simple and there is an ses (*) $0 \to P_1 \to P_2 \to P_2/P_1 \to 0$ that does not split. P_2/P_1 is simple, $\not\cong P_1$ (otherwise the ses would split). P_1 is not injective (otherwise the ses would split). P_2 and P_2/P_1 are injective. The ses (*) is a projective resolution of P_2/P_1 and an injective resolution of P_1.

	$p\dim$	$i\dim$
P_1	0	1
P_2	0	0
P_2/P_1	1	0

• Let $A = k[x]/x^2$. ${}_AA$ is indecomposable projective, and injective. $S = A/\langle x\rangle$ (onedimensional over k) is simple, neither projective nor injective, since $0 \to \underset{k}{\overset{\shortparallel}{S}} \xrightarrow{1 \mapsto x} {}_AA \xrightarrow{1 \mapsto 1} S \to 0$ is non-split, exact.

(There are no other indecomposable modules, but we cannot prove this.)

$$\begin{tikzcd}
\cdots \arrow[r] & {}_AA \arrow[rr, "1\mapsto x"] \arrow[dr] & & {}_AA \arrow[rr, "1\mapsto x"] \arrow[dr] & & {}_AA \arrow[r, "1\mapsto 1"] & S \arrow[r] & 0 \\
& & S \arrow[ur] & & S = \Omega_1(S) \arrow[ur] & & &
\end{tikzcd}$$

is an infinite projective resolution.

There is no finite projective resolution of S, as we will see later. Right now, we can already check that there is no finite finite resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to S \text{ where all } P_i \text{ are finite dimensional.}$$

Assume there is one. All P_i are sums of copies of ${}_AA$ $\Rightarrow$ their k-dimensions are even numbers. There are short exact sequences

$$0 \to \Omega_1(S) \to P_0 \to S \to 0,\quad 0 \to \Omega_2(S) \to P_1 \to \Omega_1(S) \to 0,\ \dots,\ 0 \to \overset{\Omega_n(S)}{\overset{\shortparallel}{P_n}} \to P_{n-1} \to \Omega_{n-1}(S) \to 0$$

$\dim S = 1$, $\dim P_0$ even $\Rightarrow \dim \Omega_1(S)$ odd $\Rightarrow \dim \Omega_2(S)$ odd, and so on

$\Rightarrow \dim P_n$ even (projective) and $\dim P_n = \Omega_n(S)$ odd (syzygy) ↯

In the next chapter we will do many more examples.