Now we collect properties of pullbacks and pushouts.

3.4 Lemma: Let
$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$
be a pullback square and
$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "q_1"] \\ Z \arrow[r, "q"'] & Q \end{tikzcd}$$
a pushout square.

and extend the diagrams to
$$\begin{tikzcd} \text{Kernel of } p_1 = K \arrow[d, "i"'] & \\ P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$
and
$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "q_2"] \\ Z \arrow[r, "q_1"'] & Q \arrow[d, "\bar{u}"] \\ & C = \text{cokernel of } q_2 \end{tikzcd}$$

Then $K \xrightarrow{p_2 \circ i} Y$ is the kernel of t and $Z \xrightarrow{\bar{u} \circ q_1} C$ is the cokernel of f.
In particular, there are commutative diagrams
$$\begin{tikzcd} K \arrow[r, equal] \arrow[d, "i"'] & K \arrow[d] \\ P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$
and
$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "q_2"] \\ Z \arrow[r, "q_1"] \arrow[d] & Q \arrow[d, "\bar{u}"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

(Note that these kind of diagrams play a role in the construction of the maps in Theorem 3.3.)

Proof: We do the pullback case: Using $K \xrightarrow{p_2 \circ i} Y$ gives the commutative diagram. $t \circ p_2 \circ i = s \circ p_1 \circ i = 0$ since $K \xrightarrow{i} P$ is the kernel of p_1.

We check the universal property: $K = \mathrm{Ker}(t) \xrightarrow{p_2 \circ i} Y \xrightarrow{t} Z$, giving such that $t \circ g = 0$
$$\begin{tikzcd} K \arrow[r, "p_2 \circ i"] & Y \arrow[r, "t"] & Z \\ V \arrow[u, "\exists! h"] \arrow[ur, "g"'] & & \end{tikzcd}$$
~~to~~ To show: $\exists! h: V \to K$ such that g factors, $g = (p_2 \circ i) \circ h$.

Use the property of the pullback for
$$\begin{tikzcd} V \arrow[dr, "f"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "0"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$
$\Rightarrow \exists! f: V \to P$ such that $g = p_2 \circ f$ and $0 = p_1 \circ f$.
$0 = p_1 \circ f$ implies f factors uniquely through $K = \ker(p_1)$:
$$\begin{tikzcd} V \arrow[r, "h"] \arrow[dr, "f"'] & K \arrow[d, "i"] \\ & P \end{tikzcd}$$
, $\exists! h: V \to K$

$\Rightarrow g = p_2 \circ f = p_2 \circ i \circ h$ ✓

Dually for pushout case. $\square$

Now we relate pullbacks and pushouts with short exact sequences whose middle terms are direct sums of two modules:

3.5 Lemma: A square
$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$
is a pullback diagram iff $0 \to P \xrightarrow{(p_1,p_2)} X\oplus Y \xrightarrow{\binom{s}{-t}} Z$ is exact.

A square
$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "q_2"] \\ Z \arrow[r, "q_1"'] & Q \end{tikzcd}$$
is a pushout diagram iff $X \xrightarrow{(s,t)} Y\oplus Z \xrightarrow{\binom{q_2}{-q_1}} Q \to 0$ is exact.

(Note: $t\circ p_2 = s\circ p_1$ means $\binom{s}{-t}\circ(p_1,p_2)=0$.
Starting with an ses of this type one gets both a pullback diagram and a pushout diagram. Going in the other direction, surjectivity at the right end or injectivity at the left end, is not guaranteed.)

Proof: This time we do the pushout case. We have to show that the pushout of
$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \\ Z & \end{tikzcd}$$
coincides with the cokernel of $X \xrightarrow{(s,t)} Y\oplus Z$.
The composition $\binom{q_1}{-q_2}\circ(s,t)$ is zero, like for the cokernel.
Because of uniqueness of pushout or cokernel, it is sufficient to check that one of them satisfies the universal property of the other one.
For instance: Given
$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "q_2"] \arrow[ddr, bend left, "f_1"] & \\ Z \arrow[r, "q_1"'] \arrow[drr, bend right, "f_2"'] & Q \arrow[dr, dashed, "g"] & \\ & & B \end{tikzcd}$$
with $f_1\circ s = f_2\circ t$, the composition $\binom{f_1}{-f_2}\circ(s,t)$ vanishes and therefore it uniquely factors through the cokernel. So if we choose $Q :=$ cokernel, then $\exists!\, g: Q\to B$ such that $\binom{f_1}{-f_2}$ factors through $Y\oplus Z \xrightarrow{\binom{q_2}{-q_1}} Q$ $\Rightarrow$ Q is the pushout.
The assertion on kernel and pullback is proved dually. □

3.5 tells us in particular that $P \xrightarrow{(p_1,p_2)} X\oplus Y$ in a pullback square always is injective and $\binom{q_2}{-q_1}$ in a pushout square always is surjective. When (p_1,p_2) also is surjective, i.e. an isomorphism, then by exactness, $\binom{s}{-t}$ is the zero map (check directly that for $s=t=0$, $X\oplus Y$ is the pullback). When $\binom{q_2}{-q_1}$ also is injective, then (s,t) is the zero map. (Again check directly that $Y\oplus Z$ then is the pushout)

What can we say when s or t in the pullback diagram is surjective? Or s or t in the pushout diagram is injective?

3.6 Lemma: When t in the pullback diagram is surjective, then so is p_1.
When t in the pushout diagram is injective, then so is q_2.

Proof: In the pullback case we look at the exact sequence

$$0 \to P \xrightarrow{(p_1,p_2)} X \oplus Y \xrightarrow{\begin{pmatrix} s \\ -t \end{pmatrix}} Z \quad \text{in 3.5.}$$

To show: $p_1 : P \to X$ is surjective. Let $x_0 \in X \Rightarrow s(x_0) \in Z$. $t : Y \to Z$ is surjective $\Rightarrow \exists\, y_0 \in Y : t(y_0) = s(x_0) \Rightarrow (x_0, y_0) \in \operatorname{Ker}\begin{pmatrix} s \\ -t \end{pmatrix} = \operatorname{Im}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$
$\Rightarrow x_0 \in \operatorname{Im}(p_1)$.

In the pushout case we look of course at the exact sequence

$$X \xrightarrow{(s,t)} Y \oplus Z \xrightarrow{\begin{pmatrix} q_2 \\ -q_1 \end{pmatrix}} Q \to 0 \quad \text{in 3.5}$$

To show: $q_2 : Y \to Q$ is injective. Let $y_0 \in Y$ and $q_2(y_0) = 0 \Rightarrow (y_0, 0) \in \operatorname{Ker}\begin{pmatrix} q_2 \\ -q_1 \end{pmatrix}$
$\Rightarrow (y_0, 0) = \operatorname{Im}(s,t) \Rightarrow \exists\, x_0 \in X : (y_0, 0) = (s(x_0), t(x_0))$.
t injective $\Rightarrow x_0 = 0 \Rightarrow y_0 = 0$ □

In 3.4 we have seen that pullback squares and pushout squares fit into diagrams where the top or the bottom horizontal map is the identity. Is there a converse in the sense that such diagrams always contain pullback or pushout squares? Yes, under some assumption:

3.7 Lemma: Let

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & P \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[r] \arrow[d, "s"] & 0 \\
K \arrow[r, "p_2 \circ i"'] & Y \arrow[r, "t"'] & Z \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram with exact rows.

Then the right hand square is a pullback square.

Let

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[r, "\bar{\pi} \circ q_1"] \arrow[d, "q_1"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, equal] \\
0 \arrow[r] & Y \arrow[r, "q_2"'] & Q \arrow[r, "\bar{\pi}"'] & C
\end{tikzcd}$$

be a commutative diagram with exact rows. Then the left hand square is a pushout diagram.

Proof: We do the pullback case. There is a pullback of
$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$
and we compare it with the given diagram.

Denote this pullback by
$$\begin{tikzcd} \tilde{P} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "s"] \\ Y \arrow[r, "t"'] & Z \end{tikzcd}$$
As in 3.4 we extend it to a commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\alpha) \arrow[r, "\mu"] \arrow[d, "\wr\beta"'] & \tilde{P} \arrow[r, "\alpha"] \arrow[d, "\beta"] & X \arrow[r] \arrow[d, "s"] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(t) \arrow[r, "p_2\circ\iota"'] & Y \arrow[r, "t"'] & Z \arrow[r] & 0 \end{tikzcd}$$

By assumption, t is surjective $\Rightarrow$ α is surjective, in this diagram.
$\mathrm{Ker}(t) = K$ and $\mathrm{Ker}(t) \xrightarrow{\mu\circ\beta^{-1}} \tilde{P}$ is injective

$$\Rightarrow \quad \begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\mu\circ\beta^{-1}"] \arrow[d, equal] & \tilde{P} \arrow[r, "\alpha"] \arrow[d, "\beta"] & X \arrow[r] \arrow[d, "s"] & 0 \\ 0 \arrow[r] & K \arrow[r, "p_2\circ\iota"'] & Y \arrow[r, "t"'] & Z \arrow[r] & 0 \end{tikzcd} \quad (\dagger)$$

also is a commutative diagram with exact rows. The rows are extensions of X by K or Z by K, respectively.

$\tilde{P}$ is the pullback of
$$\begin{tikzcd} & X \arrow[d, "s"] \\ Y \arrow[r, "t"'] & Z \end{tikzcd}$$
and
$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d, "s"] \\ Y \arrow[r, "t"'] & Z \end{tikzcd}$$
commutes

$\Rightarrow$ $\exists$ commutative diagram $(*)$
$$\begin{tikzcd} P \arrow[dr, "\exists\xi"] \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & \\ & \tilde{P} \arrow[r, "\alpha"] \arrow[d, "\beta"] & X \arrow[d, "s"] \\ & Y \arrow[r, "t"'] & Z \end{tikzcd}$$

We have to check that ξ is an isomorphism. Using ξ^{-1} it then follows that
$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$
has the defining property of pullback. Check that.

We are going to use the short five lemma to prove ξ is an isomorphism.
The diagram
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d] & P \arrow[r, "p_1"] \arrow[d, "\xi"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r, "\mu\circ\beta^{-1}"'] & \tilde{P} \arrow[r, "\alpha"'] & X \arrow[r] & 0 \end{tikzcd} \quad (**)$$
is commutative:

This is clear for the right hand square, by $(*)$. Moreover, $\alpha\circ\xi\circ\iota = p_1\circ\iota = 0$
$\Rightarrow$ ξ restricts to a map $K \longrightarrow \mathrm{Ker}(\alpha) \simeq K$
By assumption, p_2 restricts to the identity $K = K$, and in the diagram $(\dagger)$ β also restricts to the identity.
$\Rightarrow$ ξ restricts to the isomorphism $\beta^{-1}\circ p_2 : K \to K$.
By the short five lemma applied to $(**)$, ξ is an isomorphism.

$\square$

This finishes the second part of the proof of Theorem 3.3. The third and final part, which is starting now, is about checking that many things are well-defined and then about showing that the maps already defined are mutually inverse isomorphisms.

Recall the setup: ${}_A M$ is a module and $0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0$ is a short exact sequence whose middle term P is projective. When X is another module, we form the exact sequence $0 \to \mathrm{Hom}_A(M,X) \xrightarrow{g^*} \mathrm{Hom}_A(P,X) \xrightarrow{f^*} \mathrm{Hom}_A(K,X) \to \mathrm{Coker} f^* \to$

$\mathrm{Coker} f^*$ is to be identified with $\mathrm{Ext}^1_A(M,X)$, which does not depend on the choice of P. Therefore we better show:

__3.8 Lemma:__ Let $0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0$ and $0 \to K' \xrightarrow{f'} P' \xrightarrow{g'} M \to 0$ be short exact sequences with projective middle terms P and P'. Then

$$\mathrm{Coker} f^* \simeq \mathrm{Coker}(f')^*$$

Proof: P and P' need not be isomorphic. For example, when $P \xrightarrow{g} M$ is surjective, then $P \oplus A \xrightarrow{(g,0)} M$ also is surjective. But we can try to use projectivity to relate the different Hom-sets:

$$\begin{tikzcd} & P' \arrow[dl, dashed, "\exists \alpha"'] \arrow[d, "g'"] & \\ P \arrow[r, "g"'] & M \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & P \arrow[dl, dashed, "\exists \alpha'"'] \arrow[d] & \\ P' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$ Precomposing with α or α', respectively, defines

$\alpha^*: \mathrm{Hom}_A(P,X) \longrightarrow \mathrm{Hom}_A(P',X)$ and $(\alpha')^*: \mathrm{Hom}_A(P',X) \to \mathrm{Hom}_A(P,X)$

α and α' restrict to maps between the kernels:

$\beta: K' \to K$ and $\beta': K \to K'$ (this follows by diagram chasing)

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, shift right, "\beta'"'] & P \arrow[r, "g"] \arrow[d, shift right, "\alpha'"'] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \arrow[r, "f'"'] \arrow[u, shift right, "\beta"'] & P' \arrow[r, "g'"'] \arrow[u, shift right, "\alpha"'] & M \arrow[r] & 0 \end{tikzcd}$$

with $g \circ \alpha = g'$, $f \circ \beta = \alpha \circ f'$
$g' \circ \alpha' = g$, $f' \circ \beta' = \alpha' \circ f$

α and α' are in general, not inverse to each other, nor are β and β'!

Still it makes to look at the compositions, eg:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\beta'"] & P \arrow[r, "g"] \arrow[d, "\alpha'"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \arrow[r, "f'"] \arrow[d, "\beta"] & P' \arrow[r, "g'"] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r, "f"'] & P \arrow[r, "g"'] & M \arrow[r] & 0 \end{tikzcd}$$

and compare with

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, equal] & P \arrow[r, "g"] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r, "f"'] & P \arrow[r, "g"'] & M \arrow[r] & 0 \end{tikzcd}$$

Consider $\alpha\circ\alpha' - id_P$: By the diagram, $g\circ\alpha\circ\alpha' = g = g\circ id_P$
$\Rightarrow g\circ(\alpha\circ\alpha' - id_P) = 0 \Rightarrow \exists\, \tau: P \to K$ such that $\alpha\circ\alpha' - id_P = f\circ\tau$
Since β and β' are induced from α and α' respectively, or the commutativity of the diagram: $f\circ\beta\circ\beta' = \alpha\circ\alpha'\circ f$
$\Rightarrow f\circ(\beta\circ\beta' - id_K) = (\alpha\circ\alpha' - id_P)\circ f = f\circ\tau\circ f$
f is injective $\Rightarrow \beta\circ\beta' - id_K = \tau\circ f$, that is, τ precomposed with f
f^* is precomposing maps with f, the image of f^* is all these precomposed maps and in the cokernel of f^*, these become 0. Similarly for f' with $\tau': P' \to K'$.
This raises hope that α and α' or β and β' should induce isomorphisms between $\mathrm{Coker} f^*$ and $(\mathrm{Coker} f')^*$. To make this precise we have to look at the following diagram (and its analogue with f and f' interchanged):

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M,X) \arrow[r, "g^*"] \arrow[d, equal] & \mathrm{Hom}_A(P,X) \arrow[r, "f^*"] \arrow[d, "\alpha^*"] & \mathrm{Hom}_A(K,X) \arrow[r] \arrow[d, "\beta^*"] & \mathrm{Coker} f^* \arrow[r] \arrow[d, "\varkappa"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(M,X) \arrow[r, "(g')^*"] \arrow[d, equal] & \mathrm{Hom}_A(P',X) \arrow[r, "(f')^*"] \arrow[d, "(\alpha')^*"] & \mathrm{Hom}_A(K',X) \arrow[r] \arrow[d, "(\beta')^*"] & (\mathrm{Coker} f')^* \arrow[r] \arrow[d, "\varkappa'"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(M,X) \arrow[r, "g^*"] & \mathrm{Hom}_A(P,X) \arrow[r, "f^*"] & \mathrm{Hom}_A(K,X) \arrow[r] & \mathrm{Coker} f^* \arrow[r] & 0
\end{tikzcd}$$

Here, $\varkappa$ and $\varkappa'$ are the induced maps, making the diagram commutative.
Choose $h: K \to X$ and denote its residue class in $\mathrm{Coker} f^*$ by $\bar{h}$.
$\beta^*(h) = h\circ\beta : K' \xrightarrow{\beta} K \xrightarrow{h} X$
$(\beta')^*(\beta^*(h)) = h\circ\beta\circ\beta' : K \xrightarrow{\beta'} K' \xrightarrow{\beta} K \xrightarrow{h} X$
$\Rightarrow \varkappa'\circ\varkappa : \bar{h} \longmapsto \overline{h\circ\beta\circ\beta'} \in \mathrm{Coker} f^*$
But $h\circ(\beta\circ\beta' - id_K) = h\circ\tau\circ f \Rightarrow \overline{h\circ(\beta\circ\beta' - id_K)} = \bar{0}$ in $\mathrm{Coker} f^*$
$\Rightarrow \bar{h} = \overline{h\circ\beta\circ\beta'} \Rightarrow \varkappa'\circ\varkappa = id$ on $\mathrm{Coker} f^*$, and similarly, $\varkappa\circ\varkappa' = id_{\mathrm{Coker}(f')^*}$
$\Rightarrow \varkappa$ and $\varkappa'$ are mutually inverse isomorphisms between $\mathrm{Coker} f^*$ and $\mathrm{Coker}(f')^*$.

$\square$

Now we recall the maps $\mathrm{Ext}^1_A(M,X) \rightleftarrows \mathrm{Coker}\, f^*$ in 3.3 and check they are well-defined.

From an extension to a residue class in $\mathrm{Coker} f^*$:

Given
$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\alpha"] & E \arrow[r, "\beta"] & M \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r, "f"'] \arrow[u, "\psi"] & P \arrow[r, "g"'] \arrow[u, "\varphi"] & M \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$
form:

φ exists, since P is projective, and then ψ is the restriction of φ. The extension gets mapped to $\bar\psi \in \mathrm{Coker} f^*$.

But φ need not be unique (and in general it is far from being unique).

Claim 1: $\bar\psi$ is uniquely defined, independent of the choice of φ.

Proof: Suppose φ_1 and φ_2 are two choices for φ, that is

$$\begin{tikzcd} E \arrow[r, "\beta"] & M \\ P \arrow[u, "\varphi_1"] \arrow[r, "f"'] & M \arrow[u, equal] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \arrow[r, "\beta"] & M \\ P \arrow[u, "\varphi_2"] \arrow[r, "f"'] & M \arrow[u, equal] \end{tikzcd}$$
both are commutative diagrams.

$\Rightarrow \beta \circ (\varphi_1 - \varphi_2) = f - f = 0 \Rightarrow \exists \tau : P \to X$ such that $\varphi_1 - \varphi_2 = \alpha \circ \tau$

φ_1 and φ_2 induce $\psi_1 : K \to X$ and $\psi_2 : K \to X$ such that

$\alpha \circ \psi_1 = \varphi_1 \circ f$ and $\alpha \circ \psi_2 = \varphi_2 \circ f$, hence $\alpha \circ (\psi_1 - \psi_2) = (\varphi_1 - \varphi_2) \circ f = \alpha \circ \tau \circ f$

(Since α is injective, ψ_1 is fixed once φ_1 is so, and similarly for ψ_2.)

α injective $\Rightarrow \psi_1 - \psi_2 = \tau \circ f = f^*(\tau) \in \mathrm{Im}(f^*)$

$\Rightarrow \overline{\psi_1 - \psi_2} = \bar 0$ in $\mathrm{Coker} f^*$ $\Rightarrow \bar\psi_1 = \bar\psi_2$ in $\mathrm{Coker} f^*$ ✓

From a residue class to an extension:

Given $\bar\psi \in \mathrm{Coker}(f^*) = \mathrm{Hom}_A(K,X)/\mathrm{Im}(f^*)$, represented by $\psi \in \mathrm{Hom}_A(K,X)$ we form the diagram
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\psi"] & P \arrow[r, "g"] & M \arrow[r] & 0 \\ & X & & & \end{tikzcd}$$

and form a pushout to get

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\psi"] & P \arrow[r, "g"] \arrow[d, "\exists\varphi"] & M \arrow[r] \arrow[d, equal] & 0 \\ \xi : 0 \arrow[r] & X \arrow[r, "\exists\alpha"] & E \arrow[r, "\exists\beta"] & M \arrow[r] & 0 \end{tikzcd}$$

a commutative diagram, E = pushout of $\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "\psi"] & P \\ X & \end{tikzcd}$

The extension ξ (or rather its equivalence class) is the image of $\bar\psi$ in $\mathrm{Ext}^1(M,X)$.

φ representing $\bar\varphi$ is a choice. Suppose $\varphi' \in \mathrm{Hom}_A(K,X)$ is another choice, i.e. $\bar\varphi = \bar\varphi'$. Then $\varphi - \varphi' \in \mathrm{Im}(f^*)$, so $\exists\, \tau: P \to X$ such that $\varphi - \varphi' = \tau \circ f$.

The diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\varphi'"'] & P \arrow[r, "g"] \arrow[dl, "\tau"'] \arrow[d, "\psi'"] & M \arrow[r] \arrow[d, equal] & 0 \\ \xi: 0 \arrow[r] & X \arrow[r, "\alpha"] & E \arrow[r, "\beta"] & M \arrow[r] & 0 \end{tikzcd}$$

with $\psi' := \psi - \alpha\circ\tau$

is commutative: $\hat{}$ same E as before, not defined as pushout

$$\beta\circ\psi' = \beta\circ\psi - \underbrace{\beta\circ\alpha\circ\tau}_{0} = \beta\circ\psi = g \quad\text{and}\quad \alpha\circ\varphi' = \alpha\circ\varphi - \alpha\circ\tau\circ f = \psi\circ f - \alpha\circ\tau\circ f$$
$$= (\psi - \alpha\circ\tau)\circ f = \psi'\circ f$$

This looks good: The ses at the bottom is again ξ. But we did not follow the recipe to obtain the image of $\bar\varphi'$ in $\mathrm{Ext}^1_A(M,X)$. This requires us to form the pushout of

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "\varphi'"'] & P \\ X & \end{tikzcd}$$

By uniqueness of the pushout, it is enough to show that

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "\varphi'"'] & P \arrow[d, "\psi'"] \\ X \arrow[r, "\alpha"] & E \end{tikzcd}$$

is a pushout diagram.

This we get by invoking Lemma 3.7.

So we have shown:

<u>Claim 2:</u> ξ is uniquely defined, independent of the choice of φ.

Lemma 3.7 also helps us to prove

<u>Claim 3:</u> The two maps between $\mathrm{Ext}^1_A(M,X)$ and $\mathrm{Coker}\, f^*$ are mutually inverse to each other.

Proof: Given φ we have to form a pushout to get

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\varphi"] & P \arrow[r] \arrow[d, "\psi"] & M \arrow[r] \arrow[d, equal] & 0 \\ \xi: 0 \arrow[r] & X \arrow[r] & E \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Conversely, given an extension we have to form

$$\begin{tikzcd} \xi: 0 \arrow[r] & X \arrow[r] & E \arrow[r] & M \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r, "f"'] \arrow[u, "\varphi'"] & P \arrow[r] \arrow[u, "\psi'"] & M \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

By 3.7,

$$\begin{tikzcd} X \arrow[r] & E \\ K \arrow[r] \arrow[u] & P \arrow[u] \end{tikzcd}$$

is a pushout square for f and φ.

Since the choice of φ or φ' (or φ and ψ', respectively) does not matter, by claims 1 and 2, the two assignments are inverse to each other.

The bijections $\mathrm{Ext}^1_A(M,X) \rightleftarrows \mathrm{Coker}(f^*)$ have been established. $\mathrm{Coker}(f^*)$ naturally carries the structure of an abelian group, as a quotient of the abelian group $\mathrm{Hom}_A(K,X)$ modulo a subgroup. By transport of structure, this provides an abelian group structure on $\mathrm{Ext}^1_A(M,X)$. This turns out to be the structure we have already defined in chapter 1, using the Baer sum. If we check that it's the same zero and the same addition, then we get associativity etc for free.

The zero element in $\mathrm{Coker}(f^*)$ is represented by ψ in $\mathrm{Im}(f^*)$: $\exists \tau : P \to X$ such that $\psi = \tau \circ f$, i.e.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\psi"'] & P \arrow[r, "g"] \arrow[d, "\varphi"] \arrow[dl, "\tau"'] & M \arrow[r] \arrow[d, equal] & 0 \\ \xi : 0 \arrow[r] & X \arrow[r, "\alpha"'] & E \arrow[r, "\beta"'] & M \arrow[r] & 0 \end{tikzcd}$$

To show: ξ splits.

Use the pushout property to find a map $E \to X$:

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "\psi"'] & P \arrow[d, "\varphi"] \arrow[dd, bend left, "\tau"] \\ X \arrow[r, "\alpha"] \arrow[dr, "\mathrm{id}_X"'] & E \arrow[d, "\exists j"] \\ & X \end{tikzcd}$$

$\psi = \tau \circ f \Rightarrow$ the diagram commutes

$\Rightarrow \exists j : E \to X$ making everything commute

and satisfying $j \circ \alpha = \mathrm{id}_X \Rightarrow \xi$ splits

Check it the other way round: A split seq is mapped to $\bar{\psi} = \bar{0}$.

Concerning addition, we also start on the right hand side, with $\psi_1, \psi_2 : K \to X$

Then $\psi_1 + \psi_2$ is given as follows:

$$\begin{tikzcd} K \oplus K \arrow[r, "\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}"] & X \oplus X \arrow[d, "\nabla"] \\ K \arrow[u, "\Delta"] \arrow[r, "\psi_1 + \psi_2"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

This transfers to the Baer sum of extensions, which now is not surprising at all anymore.

This finishes the proof of theorem 3.3 $\square$