Preview: Rewriting Ext^1 as a quotient of some Hom

Adding matrices is an easy operation and forming the sum of two module homomorphisms is an operation of that sort. Our plan is to rewrite $Ext^1_A(X,Y)$ as a quotient of $Hom_A(U,V)$ for some U,V, thus "explaining" why there is an addition on $Ext^1_A(X,Y)$.
Unfortunately, we cannot ~~not~~ use X and Y as U or V, in general: For instance, X and Y can be two non-isomorphic simple modules. Then $Hom_A(X,Y)=0$ and $Hom_A(Y,X)=0$, but $Ext^1_A(X,Y)$ may be non-zero (that's why we want to understand it).
It turns out that we can keep one of the two, say Y, and then write $Ext^1(X,Y)$ as a quotient of $Hom_A(U,Y)$ for some U that we still have to find.

We start by looking at $Hom_A(X,Y)$ and what happens when we apply $Hom_A(X,-)$ or $Hom_A(-,Y)$ to the terms of a short exact sequences. Then we remember that projective modules are nice to have and that for a module M there is a projective module P and a surjective map $P \to M$, hence arises $0 \to Ker \to P \to M \to 0$.
Applying our newly acquired knowledge on Hom-sets, we suddenly stumble across the object we need to rewrite Ext^1 as a quotient of Hom.
Stating the main theorem then is easy, proving it isn't, but to do so provides us with an opportunity to better understand pullbacks and pushouts.
Having done the proof and trying to understand what happens to other sequences (not with a projective middle term), a new door opens and we start ~~a~~ to see Ext^n_A for $n \in \mathbb{N}$.