# §3. Homomorphisms, extensions and resolutions

We first have another look at homomorphisms between modules.
$R$ is a ring, $X, Y \in R\text{-Mod}$. $\mathrm{Hom}_R(X,Y)$ are the $R$-module homomorphisms from $X$ to $Y$. $\mathrm{Hom}_R(X,Y)$ always is an abelian group, $(f+g)(x) := f(x)+g(x) \in Y$.
When $K \subset Z(R)$ ($K$ a field), $\mathrm{Hom}_R(X,Y)$ is a $K$-vectorspace. In general, however, $\mathrm{Hom}_R(X,Y)$ is not an $R$-module.
Example: $R = \mathrm{Mat}(n \times n, \mathbb{C})$, $n>1$, $X = Y = \begin{pmatrix} \mathbb{C} \\ \vdots \\ \mathbb{C} \end{pmatrix}$ first column $\mathrm{End}_R(X) = \mathbb{C}$ (Schur's lemma) and $\mathbb{C}$ is not an $R$-module. why not?
There is, of course, a better output when we use a better input.
Let $S$ be another ring and ${}_R X_S$ a bimodule (in particular, $(rx)s = r(xs)$).
Claim: $\mathrm{Hom}_R({}_R X_S, {}_R Y)$ is a left $S$-module.
Check: Let $f: {}_R X_S \to {}_R Y$ be a left $R$-module homomorphism.
Then $\underset{\overset{\uparrow}{s}}{s} \cdot f: {}_R \underset{\overset{\downarrow}{x_0}}{X} \to {}_R Y$, $x_0 \mapsto f(x_0 s) \in Y$ is a left $R$-module homomorphism and $s_1(s_2 f) = (s_1 s_2) f$.
Proof of claim: $r x_0 \mapsto f(r x_0 s) = r f(x_0 s)$ where is the bimodule structure used
$\Rightarrow sf$ is in $\mathrm{Hom}_R({}_R X, {}_R Y)$
$s_1(s_2 f): x_0 \mapsto (s_2 f)(x_0 s_1) = f(x_0 s_1 s_2) = ((s_1 s_2) f)(x_0)$ ✓
Similarly, when ${}_R Y_T$ is a bimodule, then
$\mathrm{Hom}_R({}_R X, {}_R Y_T)$ is a right $T$-module.
Proof: Set $ft: x_0 \mapsto f(x_0) t \in Y$
$\Rightarrow \quad r x_0 \mapsto f(r x_0) t = r f(x_0) t$
and $f(t_1 t_2): x_0 \mapsto f(x_0) t_1 t_2$ while $(f t_1) t_2: x_0 \mapsto (f(x_0) t_1) t_2$ ✓
In particular: $\mathrm{Hom}_R({}_R X_S, {}_R Y_T)$ is an $S$-$T$ bimodule, i.e. a left $S$-module, a right $T$-module and $(sf)t = s(ft)$ ($x_0 \mapsto f(x_0 s) t$ in both cases)

How to remember these structures: $\mathrm{Hom}_R({}_R X, {}_R Y)$ are the left $R$-module homomorphisms – no further structure, the condition of being $R$-homomorphisms eats up the two $R$-structures. A right structure on $X$ moves out and becomes a left structure. A right structure on $Y$ moves out and ~~becomes~~ stays a right structure.

An interesting special case that also helps to remember the sides:
${}_R R_R$ is a bimodule over itself

For any ${}_R X$, $\mathrm{Hom}_R({}_R R_R, {}_R X) \xrightarrow[\sim]{\alpha} X$ (since $R$ is free with basis $1_R$)

$$(f: R \to X) \longmapsto f(1)$$

The right $R$-module structure on $R$ provides a left $R$-module structure on $\mathrm{Hom}_R({}_R R, X)$ as above: $(rf): r' \longmapsto f(r'r)$

$$1 \longmapsto f(1r) = f(r1) = r f(1)$$

$\Rightarrow \alpha(rf) = r\alpha(f) \Rightarrow \alpha$ is a left $R$-module isomorphism

$${}_R\mathrm{Hom}_R({}_R R_R, {}_R X) \xrightarrow[\sim]{\alpha} {}_R X$$

(Everything else would be a surprise, wouldn't it?)

Every module is a bimodule: Let ${}_R X$ be a left module and $E := \mathrm{End}_R(X)$ its endomorphism ring. Define $\underset{X \ni}{x_0} \underset{\in E^{op}}{f} := f(x_0)$.

$\Rightarrow (x_0 f) g = g(x_0 f) = g(f(x_0)) = (x_0)(g \circ f) = (x_0)(f * g)$ where $*$ is the multiplication in the opposite ring $E^{op}$. This gives an $E^{op}$-structure on $X$.

The bimodule condition is: $\underset{f(rx_0)}{\underset{\shortparallel}{(rx_0)f}} \overset{!}{=} \underset{r f(x_0)\ \checkmark}{\underset{\shortparallel}{r(x_0 f)}}$

$\Rightarrow \mathrm{Hom}_R({}_R X_S, {}_R Y_T)$ is an $S$-$T$-bimodule, and in particular

$\mathrm{Hom}_R({}_R X, {}_R Y)$ is an $\mathrm{End}_R(X)^{op}$-$\mathrm{End}_R(Y)^{op}$-bimodule.

(This is one of the places, where one may regret the convention $fg$ = "first apply $g$ then $f$")

When we fix ${}_R X$, we can plug many ${}_R Y$ into $\mathrm{Hom}_R({}_R X, -)$ and get many Hom-spaces (or Hom-groups, when there is no $K$ around). How are these related when $Y$ varies? More precisely, when $\alpha: Y_1 \to Y_2$ is a homomorphism, how are $\mathrm{Hom}_R(X, Y_1)$ and $\mathrm{Hom}_R(X, Y_2)$ related?

Choose an element $f \in Hom_R(X,Y_1)$, i.e. $f: X \to Y_1$. Then we can post compose with $\alpha$ to get
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\alpha\circ f"'] & Y_1 \arrow[r, "\alpha"] & Y_2 \end{tikzcd} \in Hom_R(X,Y_2).$$

$\rightsquigarrow \alpha: Y_1 \to Y_2$ induces a map $Hom_R(X,Y_1) \to Hom_R(X,Y_2)$

notation: $\alpha_*$ or $Hom(-,\alpha)$ or $Hom(X,\alpha)$

Of course, we can do something similar with $\beta: X_1 \to X_2$ and fixed $Y$:

(pre-compose)
$$\begin{tikzcd} X_1 \arrow[r, "g\beta"] \arrow[d, "\beta"'] & Y \arrow[d, equal] \\ X_2 \arrow[r, "g"] & Y \end{tikzcd}$$
given

But now $\beta^*: Hom_R(X_2,Y) \to Hom_R(X_1,Y)$, this goes in the opposite direction. (Therefore we place the asterisk differently.)

We call $\alpha_*$ <u>covariant</u> (keeps the direction of arrows) and $\beta^*$ <u>contravariant</u> (reverses the direction).

This provides us with a very interesting option to look at short exact sequences again: Given an ses $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ and another module $M$, we can apply $Hom_R(M,-)$ or $Hom_R(-,M)$ to this ses (to the modules and the maps) and see what happens to exactness.

Let's first look at a split sequence: $0 \to X \xrightarrow{(1,0)} X\oplus Z \xrightarrow{(0,\pi)} Z \to 0$.

Then $Hom_R(X\oplus Z, M) = Hom_R(X,M) \oplus Hom_R(Z,M)$ what are the isomorphisms?

and $Hom_R(M, X\oplus Z) = Hom_R(M,X) \oplus Hom_R(M,Z)$

imply $0 \to Hom(Z,M) \xrightarrow{g^*} Hom(Z,M) \oplus Hom(X,M) \xrightarrow{f^*} Hom(X,M) \to 0$ exact

and $0 \to Hom(M,X) \xrightarrow{f_*} Hom(M,X) \oplus Hom(M,Z) \xrightarrow{g_*} Hom(M,Z) \to 0$ exact.

So, the short exact sequence gets turned into new short exact sequences. This is not true in general:

Let $K$ be a field and $A = K[x]/(x^2)$, a two-dimensional $K$-algebra. $\langle x \rangle$ is a one-dimensional ideal, $A/\langle x\rangle = K$ is simple.

Check that $\langle x \rangle = K$ as $A$-module and that up to isomorphism this is the only simple $A$-module.

$\Rightarrow$ There is a short exact sequence

$$0 \to k \cong \langle x \rangle \xrightarrow{\text{incl}} k[x]/x^2 \xrightarrow{\text{quot}} k \to 0$$

Apply $\mathrm{Hom}_A(k,-)$ (covariant) and get

$$\underset{\substack{\| \\ 0}}{\mathrm{Hom}_A(k,0)} \xrightarrow{0_*} \underset{\substack{\cong \\ k}}{\mathrm{Hom}_A(k,k)} \xrightarrow{\text{incl}_*} \underset{\substack{\cong \\ k}}{\mathrm{Hom}_A(k, k[x]/x^2)} \xrightarrow{\text{quot}_*} \underset{\substack{\cong \\ k}}{\mathrm{Hom}_A(k,k)} \xrightarrow{0_*} \underset{\substack{\| \\ 0}}{\mathrm{Hom}_A(k,0)}$$

$$k \xrightarrow{\sim} k \xrightarrow{\text{incl}} k[x]/x^2 \qquad \underbrace{k \to k[x]/x^2 \to k}_{0}, \quad \lambda \mapsto \lambda \cdot \bar{x} \mapsto 0$$

$\rightsquigarrow 0 \to k \xrightarrow{\sim} k \xrightarrow{0} k \to 0$, not exact
$\qquad\qquad\qquad \underset{\ker(0) \neq \mathrm{im}(0)}{\|}$

Apply $\mathrm{Hom}_A(-,k)$ (contravariant) and get

$$\underset{\substack{\| \\ 0}}{\mathrm{Hom}_A(0,k)} \to \underset{\substack{\cong \\ k}}{\mathrm{Hom}_A(k,k)} \xrightarrow{\text{quot}^*} \underset{\substack{\cong \\ k}}{\mathrm{Hom}_A(k[x]/x^2,k)} \xrightarrow{\text{incl}^*} \underset{\substack{\cong \\ k}}{\mathrm{Hom}_A(k,k)} \to \underset{\substack{\cong \\ 0}}{\mathrm{Hom}_A(\{0\},k)}$$

$$k[x]/x^2 \to k \to k \qquad \underbrace{k \to k[x]/x^2 \to k}_{0}$$

Again, not exact.
In both cases, the problem occurs at the same place and elsewhere exactness works. This reflects the general situation:

<u>3.1 Theorem:</u> Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence of $R$-modules and $M$ also an $R$-module. Then the sequences

$0 \to \mathrm{Hom}_R(M,X) \xrightarrow{f_*} \mathrm{Hom}_R(M,Y) \xrightarrow{g_*} \mathrm{Hom}_R(M,Z)$ and

$0 \to \mathrm{Hom}_R(Z,M) \xrightarrow{g^*} \mathrm{Hom}_R(Y,M) \xrightarrow{f^*} \mathrm{Hom}_R(X,M)$ are exact.

We say: $\mathrm{Hom}_R(M,-)$ and $\mathrm{Hom}_R(-,M)$ are <u>left exact</u> (for all $M$).

Proof: Let us check exactness of the second sequence:

$g^*$ is injective: Let $\alpha: Z \to M$, then $g^*(\alpha): Y \xrightarrow{g} Z \xrightarrow{\alpha} M$. Assume $g^*(\alpha) = 0$.
Want: $\alpha = 0$. For $z_0 \in Z$ $\exists y_0 \in Y$: $z_0 = g(y_0)$ as $g$ is surjective.
$\Rightarrow \alpha(z_0) = \alpha(g(y_0)) = g^*(\alpha)(y_0) = 0$ ✓

$f^* \circ g^* = 0$: $(f^* \circ g^*)(\alpha)$: $\underbrace{X \xrightarrow{f} Y \xrightarrow{g} Z}_{f^*(g^*(\alpha))} \xrightarrow{\alpha} M$. But $g \circ f = 0$. Hence $f^* g^* = 0$.

This implies: $\mathrm{Im}(g^*) \subset \mathrm{Ker}(f^*)$. ✓

$\mathrm{Ker}(f^*) \subset \mathrm{Im}(g^*)$: Let $\beta: Y \to M$ be in $\mathrm{Ker}(f^*)$, i.e. $X \xrightarrow{f} Y \xrightarrow{\beta} M$, $\mathrm{Im}(f) \subset \mathrm{Ker}(\beta)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "0"'] & Y \arrow[r, "\beta"] & M \end{tikzcd}$$

By assumption, $\mathrm{Im}(f) = \mathrm{Ker}(g)$

$\Rightarrow$ $\beta$ factors through the cokernel of $f$, which means it factors through $g$:

$\exists \mu: Z \to M$ such that $\beta = \mu \circ g = g^*(\mu) \in \mathrm{Im}(g^*)$ ✓

Exactness of the first sequence can be shown in a similar way □

Sometimes, $\mathrm{Hom}_A(M,-)$ or $\mathrm{Hom}_A(-,N)$ send ses to ses, for instance when the given ses splits. For which $M$ or $N$ is the result always exact?

3.2 Theorem: The functor $\mathrm{Hom}_A(M,-)$ is exact, that is, it sends each short exact sequence to a short exact sequence $\Leftrightarrow$ ${}_AM$ is projective.

The functor $\mathrm{Hom}_A(-,N)$ is exact $\Leftrightarrow$ ${}_AN$ is injective.

Proof: Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence. $\mathrm{Hom}_A(M,-)$ turns it into the sequence $0 \to \mathrm{Hom}_A(M,X) \xrightarrow{f_*} \mathrm{Hom}_A(M,Y) \xrightarrow{g_*} \mathrm{Hom}_A(M,Z)$

The question is when is $g_*$ surjective. $g_*$ means $\forall \alpha: M \to Z \ \exists \beta: M \to Y$ such that $\alpha = g_*(\beta) = g \circ \beta$. In a diagram

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \arrow[r] & 0 \\ & M \arrow[ul, dashed, "\exists\beta"] \arrow[u, "\forall\alpha"'] & \end{tikzcd}$$

By definition 2.3 this is the definition of $M$ being projective.

( This uses that any $Y \xrightarrow{g} Z \to 0$ can be completed to a ses $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, for instance by choosing $(X,f) = (\mathrm{Ker}(g), \mathrm{incl})$.)

The second statement has a similar proof. □

When $g_*$ or $f^*$ are not surjective, the cokernel is non-zero and may contain interesting information (in addition to being non-zero, which is interesting, too). We start with particular ses, which will turn out to provide very interesting and relatively accessible information:

Let $M$ be any $A$-module ($A$ any ring or algebra). Choose a projective module $P$ such that there is a surjection $g: P \xrightarrow{g} M \to 0$. Complete this to a short exact sequence using $K = \mathrm{Ker}(g)$ and $f$ the inclusion: $0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0$

Let $X$ be any $A$-module and apply $Hom_A(-,X)$ to these, and get

$$0 \to Hom_A(M,X) \xrightarrow{g^*} Hom_A(P,X) \xrightarrow{f^*} Hom_A(K,X) \to ? = Coker(f^*) \to 0$$

We want to understand $? = Coker(f^*)$. All Hom-sets are abelian groups (or even $F$-spaces when $A$ is an $F$-algebra, for instance over a field $F$).
$f^*$ is additive (and $F$-linear in the $F$-algebra case) $\Rightarrow$ $Coker(f^*)$ is defined and an abelian group (or an $F$-space), where addition comes from addition of maps in $Hom_A(K,X)$.
There is another structure present: $Hom_A(K,X)$ and $Hom_A(P,X)$ have a right $End_A(X)^{op}$-module structure, which $f^*$ respects:

$$\begin{tikzcd} P \arrow[r, "\alpha"] & X \arrow[d, "\beta"'] \\ & X \end{tikzcd} \quad \overset{f^*}{\longmapsto} \quad \begin{tikzcd} K \arrow[r, "f"] \arrow[rr, bend right, "f^*(\alpha)"'] & P \arrow[r, "\alpha"] & X \arrow[d, "\beta"] \\ & & X \end{tikzcd} \qquad f^*(\beta \circ \alpha) = \beta \circ f^*(\alpha)$$

Now comes a huge surprise:

<u>3.3 Theorem</u>: There is an isomorphism of abelian groups (or vectorspaces), even of $End_A(X)^{op}$-modules:

$$Ext_A^1(M,X) \simeq Coker(f^*) = Hom_A(K,X)/\underbrace{Im(f^*)}_{=f^*(Hom_A(P,X))}$$

Suddenly, a recipe for computing $Ext_A^1(M,X)$ has appeared, and an explanation for the abelian group structure of $Ext_A^1(M,X)$: The right hand side, $Coker(f^*)$ has such a structure, and addition is just addition of morphisms.
$Coker(f^*)$ looks much more accessible than $Ext_A^1(M,X)$, and we will see that this is true: We will have to find a sequence $0 \to K \to P \to M \to 0$, compute the homomorphisms into $X$ and then form the quotient $Hom_A(K,X)/Im(f^*)$.
Each step will require some work, but in principle it is doable.

The right hand side depends on the choice of $P \xrightarrow{g} M$, while the left hand side doesn't. We will have to check what changing $P \xrightarrow{g} M$ means.
The proof of 3.3 will require quite some work, including understanding pullbacks and pushouts better.

Before embarking on the proof we look at an example of $\mathbb{Z}$-modules:

$A = \mathbb{Z}$, $M = \mathbb{Z}/3\mathbb{Z}$. Choose $P = \mathbb{Z}$ (free, hence projective), $g: P \to M$, $n \mapsto \bar{n}$ the residue class map $\Rightarrow \ker(g) = 3\mathbb{Z}$

$$\rightsquigarrow 0 \to \mathbb{Z} \xrightarrow[\text{incl}]{f} \mathbb{Z} \xrightarrow[\text{proj}]{g} \mathbb{Z}/3\mathbb{Z} \to 0$$

$\mathbb{Z} \cong 3\mathbb{Z} = K$, $\mathbb{Z} = P$, $\mathbb{Z}/3\mathbb{Z} = M$, $f = 3\cdot$

$f: 1 \mapsto 3$

We choose $X_1 = \mathbb{Z}/2\mathbb{Z}$ and afterwards $X_2 = \mathbb{Z}/3\mathbb{Z}$.

$X_1 = \mathbb{Z}/2\mathbb{Z} \rightsquigarrow$ applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$ to the above sequence gives

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{g^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{f^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{coker} f^* \to 0$$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$; $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \xrightarrow{f^*} \mathbb{Z}/2\mathbb{Z}$; $\mathrm{coker} f^* = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$

$\bar{1} \mapsto \bar{0}$ : 0 map

$\bar{1} \mapsto \bar{1} \Rightarrow \bar{2} \mapsto \bar{0}$

$\Rightarrow \underbrace{2\cdot\bar{2}}_{=\bar{1}} \mapsto \bar{0}$

$1 \mapsto \bar{0}$ or $1 \mapsto \bar{1}$ $\qquad$ $1 \mapsto \bar{0}$ or $1 \mapsto \bar{1}$

$f^*(\alpha: \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}) = \alpha \circ f: \mathbb{Z} \xrightarrow{3\cdot} \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z}$

$1 \mapsto 3 \mapsto 3\alpha(1) = \alpha(1)$

$\Rightarrow f^*$ isomorphism

$\Rightarrow \mathrm{Coker}(f^*) = 0$,

which confirms that each extension of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$ splits.

$X_2 = \mathbb{Z}/3\mathbb{Z}$: apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/3\mathbb{Z}) \rightsquigarrow$

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{g^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{f^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{Coker} f^* \to 0$$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$, $\alpha$; $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$; $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$; $\mathrm{Coker} f^* = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$

$\bar{1} \mapsto \alpha(\bar{1})$

$g^*(\alpha)$:

$1 \overset{g}{\mapsto} \bar{1} \mapsto \alpha(\bar{1})$

$g^*$ injective

$\Rightarrow$ isomorphism

sequence exact $\Rightarrow$ $\mathrm{Coker} f^* \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$

$\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \simeq \mathbb{Z}/3\mathbb{Z}$

We knew already that there is a non-split extension.

Now we know there are three equivalence classes of extensions:

The split extension $0 \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$ represents one class.

$0 \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$ represents another class.
$\bar{1} \mapsto \bar{3}$ $\quad \bar{1} \mapsto \bar{1}$

Two more candidates: $0 \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$
$\bar{1} \mapsto \bar{6}$ $\quad \bar{1} \mapsto \bar{1}$

and $0 \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$
$\bar{1} \mapsto \bar{3}$ $\quad \bar{1} \mapsto \bar{2}$

} check: these two are equivalent to each other, but not to the previous sequences

The proof of 3.3 will consist of three parts:
First we define maps $\mathrm{Ext}^1_A(M,X)$ to $\mathrm{Coker}(f^*)$ and back. This is the part we really need when we compute extensions from $\mathrm{Coker}(f^*)$.
The constructions, and already $\mathrm{Coker}(f^*)$ itself, depend on many choices. Thus we have verify that various things are well-defined, which means independent of the choices. This checking and verifying that the maps of the first part are mutually converse and have the desired properties, is the third part of the proof.
In between, there will be a second part of the proof, collecting information needed in the third part, in particular about pullbacks and pushouts.

## The maps $\mathrm{Ext}^1_A(M,X) \rightleftarrows \mathrm{Coker}(f^*)$ in Theorem 3.3

Let $\xi: 0 \to X \xrightarrow{\alpha} E \xrightarrow{\beta} M \to 0$ be an extension of $M$ by $X$.
Extend to this diagram:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\alpha"] & E \arrow[r, "\beta"] & M \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r, "f"] \arrow[u, "\exists \psi"] & P \arrow[r, "g"] \arrow[u, "\exists \varphi"] & M \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

$\varphi: P \to E$ exists because $P$ is projective and has the lifting property.
$\Rightarrow g = \beta \circ \varphi \Rightarrow \underbrace{g \circ f}_{0} = \beta \circ \varphi \circ f \Rightarrow \varphi \circ f$ factors through $\mathrm{Ker}(\beta) = \alpha(X) \cong X$
and therefore also through $X \Rightarrow \psi: K \to X$ exists such that the diagram commutes.
$\psi \in \mathrm{Hom}_A(K,X)$, $\bar{\psi} \in \mathrm{Coker}(f^*)$ is the residue class of $\psi$. This is the image of $\xi$.

The map in the opposite direction is as follows:

We are given an element $\bar{\psi}$ represented by $\psi: K \to X$.

$$\operatorname{Coker}(f^*) = \operatorname{Hom}_A(K,X)/\operatorname{Im}(f^*)$$

So:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\psi"] & P \arrow[r, "g"] & M \arrow[r] & 0 \\ & X & & & \end{tikzcd}$$

and we want an extension

$$0 \to X \to ? \to M \to 0$$

Complete the diagram by taking a pushout

$$\begin{tikzcd} & 0 \arrow[r] & K \arrow[r] \arrow[d, "\psi"] & P \arrow[r] \arrow[d, "\exists \varphi"] & M \arrow[r] \arrow[d, equal] & 0 \\ \xi: & 0 \arrow[r] & X \arrow[r] & E \arrow[r] \arrow[d, equal] & M \arrow[r] & 0 \\ & & & \text{pushout} & & \end{tikzcd}$$

The equivalence class of $\xi$ in $\operatorname{Ext}^1_A(M,X)$ is the image of $\bar{\psi}$.

These maps are quite natural, but there is much to be checked later on:

How do the maps depend on the choice of $P \xrightarrow{g} M$?

How does the image of $\xi$ depend on the choice of $\varphi: P \to E$?

How does the image of $\bar{\psi}$ depend on the choice of $\psi$?

And, of course, are the two maps mutually inverse to each other?