Exercises on monomorphisms and epimorphisms, and (co) products

(1) A homomorphism $f: X \to Y$ of left R-modules is a <u>monomorphism</u> (or a monic morphism or a mono) $:\Leftrightarrow$ f is left cancellative, that is, for all left R-modules Z and $g_1, g_2: Z \to X$ module homomorphisms:

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

Prove: (a) f is mono $\Leftrightarrow$ f is injective $\Leftrightarrow$ kernel(f) is 0

(b) $\forall\, h: M \to N$, $i: K \to M$ kernel of h: i is mono

(c) If f is left invertible ($\exists\, l: Y \to X$: $l \circ f = id_X$), then f is mono.

(Try in all cases to avoid using properties of modules if possible.)

(d) Now we change from R-modules to divisible abelian groups: Show that the quotient homomorphism $\pi: \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a surjective mono, but not an isomorphism.

(2) A homomorphism $f: X \to Y$ of left R-modules is an <u>epimorphism</u> (or an epic morphism or an epi) $:\Leftrightarrow$ f is right cancellative, that is, for all left R-modules Z and $g_1, g_2: Y \to Z$: $g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$

~~Prove~~ Formulate and prove (a'), (b') and (c').

(d') Now we change from R-modules to rings: Show that the inclusion $i: \mathbb{Z} \hookrightarrow \mathbb{Q}$ of rings is an injective epimorphism, but not an isomorphism.

(3) An <u>equaliser</u> of $f: X \to Y$ and $g: X \to Y$ is $h: E \to X$ such that $f \circ h = g \circ h$ and $\forall\, Z$

$$\begin{tikzcd} Z \arrow[d, "\exists ! u"'] \arrow[dr, "\forall i"] & & \\ E \arrow[r, "h"'] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

Show that equalisers exist for R-module homomorphisms, and are unique.

Define coequalisers and prove similar statements.

(4) A <u>product</u> of a family of objects X_i $(i \in I)$ is an object X with morphisms $\pi_i : X \to X_i$ for $i \in I$ such that

$$\begin{tikzcd} & X \arrow[d, "\pi_i"] \\ \forall Y \arrow[ur, dashed, "\exists! f"] \arrow[r, "\forall f_i"'] \arrow[ur, phantom, "\circlearrowright" near end, shift right=3] & X_i \end{tikzcd}$$

Show that products exist for sets and for R-modules.

(5) A <u>coproduct</u> of a family of objects X_j $(j \in I)$ is an object X with morphisms $\iota_j : X_j \to X$ for $j \in I$ such that

$$\begin{tikzcd} & X \arrow[dl, dashed, "\exists! f"'] \\ \forall Y \arrow[ur, phantom, "\circlearrowleft" near end, shift right=3] & X_j \arrow[u, "\iota_j"'] \arrow[l, "\forall f_j"] \end{tikzcd}$$

(6) Write a pushout in terms of a coproduct and a coequaliser.
Write a pullback in a similar way.