Exercises on projective and injective modules

(1) Let R be a ring and P an R-module. Show that P is projective

$\iff$ $\exists\, I$, a set, $\{a_i \mid i\in I\}$ a subset of P and $\{f_i \in \mathrm{Hom}_R(P,R) \mid i \in I\}$

such that $\forall x \in P$: $\{i \in I : f_i(x) \neq 0\}$ is finite

and $x = \sum\limits_{i\in I} f_i(x)\, a_i$

(2) Let R be a ring and I an R-module. Show that I is injective

$\iff$ $\forall$ left ideals L in R and for all diagrams

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook, "\mathrm{incl}"] \arrow[d, "\forall f"'] \arrow[dr, phantom, "\circlearrowright"] & R \arrow[dl, dashed, "\exists g"] \\ & I & \end{tikzcd}$$

(f, g: module homomorphisms)

(You may use Zorn's lemma)

(3) Let R be a principal ideal domain and I an R-module satisfying $\forall r \in R, r \neq 0$: $\underbrace{rI}_{=\{r\cdot x :\, x \in I\}} = I$. Show that I is injective.

Give examples of injective modules for $R = \mathbb{Z}$, $R = K[x]$ or general R.

Are $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ injective $\mathbb{Z}$-modules?