Free modules, projective modules and an undecidable problem

(This is bonus material, not to be examined.)
When $R = K$ is a field, then all R-modules are free and all ses split.
In particular, projective $\Leftrightarrow$ free.
The latter equivalence occurs more generally (at least for finitely generated modules):
Let R be a principal ideal domain. Then a submodule of a free module is free.
For finitely generated modules, this is part of the elementary divisors theorem (see eg Theorem 5.16 in last semester's Algebra II). The general case then follows by using Zorn's lemma.
$\Rightarrow$ For R PID, projective $\Leftrightarrow$ free.
For $R = K[X_1, ..., X_n]$ (any $n \in \mathbb{N}$), any finitely generated projective module is free. This is a difficult theorem by Quillen and Suslin (see eg. S. Lang, Algebra, XXI, 3.7).

Now we fix $R = \mathbb{Z}$. A free (or projective) abelian group M satisfies:
$\mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) = 0$.
Whitehead problem: Does $\mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) = 0$ imply that M is free?

When M is countable, this is true as show by Karl Stein in 1951, working in analysis.
The general answer is unexpected:
Theorem (Saharon Shelah, 1974/77/80): In Zermelo-Fraenkel set theory, assuming the axiom of choice, the Whitehead problem is undecidable.
More precisely, the answer is yes in the constructible universe.
The answer is no, assuming Martin's axiom (and one may assume the continuum hypothesis or its negation).
(References: S. Shelah, Infinite abelian groups, Whitehead problem and some constructions, Israel J. of Math 18, 1974, 243-256.
P. Eklof, Whitehead's problem is undecidable. American Math Monthly, 83, 1976, 775-788)