§ 16. Repetitive algebras

Let A be a finite-dimensional algebra over a field k. To simplify notation A is assumed to be basic. We consider the finite dimensional left A-modules, i.e. the category A-mod. $\mathrm{Hom}_k(-,k) =: D$ is the k-duality, turning left modules into right modules and vice versa. $Q := DA$ is injective, $\mathrm{add}\, Q = A$-inj.

To A we assign an infinite dimensional algebra:

16.1 Definition: The <u>repetitive algebra</u> $\hat{A}$ of A has underlying vector space

$\hat{A} = (\bigoplus_{i\in\mathbb{Z}} A) \oplus (\bigoplus_{i\in\mathbb{Z}} Q)$ with elements $(a_i, \varphi_i)_{i\in\mathbb{Z}}$, $a_i \in A$, $\varphi_i \in Q$, almost all a_i and φ_i being zero.

Multiplication is $(a_i, \varphi_i)_i \cdot (b_i, \psi_i)_i := (a_i b_i, a_{i+1}\psi_i + \varphi_i b_i)_i$

A more suggestive way to write $\hat{A}$ is as a doubly infinite matrix algebra

$$\begin{pmatrix} \ddots & & & & \\ & A_{i-1} & & 0 & \\ & Q_{i-1} & A_i & & \\ & & Q_i & A_{i+1} & \\ & 0 & & & \ddots \end{pmatrix}_{i\in\mathbb{Z}}$$

all $A_i = A$, $Q_i = Q$ (with indices just for book keeping)

$\hat{A}$ satisfies all the usual axioms of an algebra except one; it does not have a unit element, as the matrix $\begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$ is not in $\hat{A}$.

More formally, multiplication is induced from multiplication $A \otimes_A A \to A$, the module structures $A \otimes_A Q \to Q$ and $Q \otimes_A A \to Q$, and the zero map $Q \otimes_A Q \to Q$.

The elements $\begin{array}{c} \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} =: e_i \\ \uparrow \\ i \end{array}$ are idempotents $\Rightarrow$ an $\hat{A}$-module M decomposes as a vector space into $M = \underbrace{\bigoplus_{i\in\mathbb{Z}} M_i}_{\text{view as column vectors } \begin{pmatrix} \vdots \\ M_{i-1} \\ M_i \\ M_{i+1} \\ \vdots \end{pmatrix}}$. Here, M_i is an A_i-module and in addition $Q_{i-1} \otimes_A M_i \to M_{i+1}$ is the action of Q_{i-1} on M_i.

This structure map $Q \otimes_A M_i \to M_{i+1}$ is denoted by f_i. Since $Q \otimes Q$ is zero, it must satisfy $f_{i+1} \circ (1 \otimes f_i) = 0$:

$$\begin{tikzcd} Q \otimes_A Q \otimes_A M_i \arrow[r] \arrow[rr, bend right, "0"'] & Q \otimes_A M_{i+1} \arrow[r] & M_{i+2} \end{tikzcd}$$

$\Rightarrow$ An $\hat{A}$-module has the form $M = (M_i, f_i)_{i \in \mathbb{Z}}$ where the M_i are A-modules and the f_i are A-linear maps $f_i : Q \otimes_A M_i \to M_{i+1}$ satisfying $f_{i+1} \circ (1 \otimes f_i) = 0$.

Example: Let X be an A-module, $i \in \mathbb{Z}$ and set $M_j := \begin{cases} X, & j = i \\ 0, & j \neq i \end{cases}$ and $f_j = 0$ for all j.

Notation: $M = (M_i, f_i)$ is written as $\cdots M_{-2} \overset{f_{-2}}{\sim} M_{-1} \overset{f_{-1}}{\sim} M_0 \overset{f_0}{\sim} \cdots$ (where the symbol has no deeper meaning, we just should not use an arrow here).

A morphism $h : M = (M_i, f_i)_{i \in \mathbb{Z}} \to N = (N_i, g_i)_{i \in \mathbb{Z}}$ is a family of A-linear maps $h_i : M_i \to N_i$ such that for all $i \in \mathbb{Z}$ the square

$$\begin{tikzcd} Q \otimes_A M_i \arrow[r, "f_i"] \arrow[d, "1 \otimes h_i"'] \arrow[dr, phantom, "\circlearrowright"] & M_{i+1} \arrow[d] \\ Q \otimes N_i \arrow[r] & N_{i+1} \end{tikzcd}$$

commutes.

The matrix $1_i = \begin{array}{cc} \begin{pmatrix} \ddots & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow i \\ \\ \\ \end{array} \\ \begin{array}{c} \uparrow \\ i \end{array} & \end{array}$ is an idempotent in $\hat{A}$, which is a sum of primitive pairwise orthogonal idempotents in the same way as the unit 1_A of A is.

Multiplication by 1_i is an exact functor that is naturally isomorphic to $\mathrm{Hom}_{\hat{A}}(\hat{A} 1_i, -)$ and to $1_i \hat{A} \otimes_{\hat{A}} -$ ($\hat{A}$ has no unit, thus the tensor product has to be redefined)

$\Rightarrow$ $\hat{A} 1_i$ is a projective module and $\hat{A} = \bigoplus_{i \in \mathbb{Z}} \hat{A} 1_i$ (which indicates that the family $\{1_i\}_{i \in \mathbb{Z}}$ is a good substitute for the missing unit element).

$\hat{A} 1_i = \begin{pmatrix} & | & \\ - & A & - \\ & Q & \\ & | & \end{pmatrix}$ and $1_i \hat{A} = \begin{pmatrix} & | & \\ - & A \; Q & \\ & | & \end{pmatrix}$. Both are finite dimensional over k.

When $M = (M_i, f_i)$ is finite dimensional, we can apply k-duality D and $D(DM) \simeq M$ (natural isomorphism) as usual. For infinite dimensional M this fails. However, we can decompose $M = \bigoplus M_i$ and then dualise "locally" by setting $DM := \bigoplus DM_i$. Then $D^2(M) \simeq M$. This works as usual.

Since $Q = D(A)$, the dual of the projective right $\hat{A}$-module $1_i \cdot \hat{A} = \left(\cdots AQ \cdots \right)$ is isomorphic to the left $\hat{A}$-module $\hat{A} \cdot 1_i$, which is projective.
Result: $\hat{A}$ is a Frobenius algebra, projective and injective modules coincide (Again one has to modify a definition, that of a Frobenius algebra.)
Slightly modifying the proof of theorem 12.14 it follows that:
The stable category $\hat{A}$-$\underline{\text{mod}}$ is a triangulated category with shift $\Sigma = \Omega^{-1}$!

Here $\hat{A}$-mod refers to finitely generated $\hat{A}$-modules. Since $\hat{A} \cdot 1_i$ is finite dimensional, for each i, finitely generated implies finite dimensional. This implies that $\hat{A}$-mod is an abelian category, since it is closed under submodules and quotients.

Now we are going to investigate the triangulated category $\hat{A}$-$\underline{\text{mod}}$.

<u>16.2 Lemma:</u> The composition $A\text{-mod} \longrightarrow \hat{A}\text{-mod} \longrightarrow \hat{A}\text{-}\underline{\text{mod}}$ is fully faithful.
$$M \longmapsto M = M_0 \qquad f \mapsto \underline{f}$$

Proof: Let $X, Y \in A$-mod, $f : X \to Y$ a module homomorphism. In $\hat{A}$-mod, f becomes

$$\begin{tikzcd} Q \otimes X_{-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X = X_0 \arrow[d, "f"] \\ Q \otimes Y_{-1} \arrow[r] & Y = Y_0 \end{tikzcd}$$

where $X_{-1} = Y_{-1} = 0$, so it's just f again.
$\mathrm{Hom}_{\hat{A}\text{-}\underline{\text{mod}}}(X, Y)$ is a quotient, which implies full.

What does it mean for f to become 0 in $\hat{A}$-$\underline{\text{mod}}$? Then it factors through an injective $\hat{A}$-module, hence through the injective envelope of X in $\hat{A}$-mod:

$f = f_2 \circ f_1$ with

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f_1"] \\ Q \otimes_A I \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & I \arrow[d, "f_2"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

(here, I may be chosen as injective envelope of X).
As $Q = DA$, $I \in \mathrm{add}\, Q$. Moreover,
$Q \otimes_A Q = DA \otimes_A DA \simeq D(A \otimes_A A) \simeq DA = Q$.

$\Rightarrow$ $Q \otimes Q \to Q$ would be an isomorphism, hence $Q \otimes I \to I$ is an isomorphism, too.
$\Rightarrow f_2 = 0 \Rightarrow f = f_2 \circ f_1 = 0$. This shows faithful. $\square$

16.3 Proposition: Let X and Y be A-modules and $i \in \mathbb{Z}$. Then

$$\underline{\mathrm{Hom}}(X, Y[i]) \simeq \mathrm{Ext}^i_A(X,Y)$$

(where $\underline{\mathrm{Hom}}$ is taken in the stable category $\hat{A}$-$\underline{\mathrm{mod}}$)

Proof: Consider Y as an $\hat{A}$-module via the embedding A-mod $\to \hat{A}$-mod. As $\hat{A}$-module, Y has an injective resolution $0 \to Y \xrightarrow{\mu} I_0(Y) \xrightarrow{d_0} I_1(Y) \xrightarrow{d_1} I_2(Y) \xrightarrow{d_2} \dots$. It also has an injective resolution as an A-module $0 \to Y \xrightarrow{\mu_0} J_0(Y) \xrightarrow{e_0} J_1(Y) \xrightarrow{e_1} J_2(Y) \xrightarrow{e_2} \dots$ and $J_i(Y)$ is the degree 0-part of $I_i(Y)$, thus $\mu_0, e_0, e_1, \dots$ are the degree 0-terms of $\mu, d_0, d_1, \dots$

Since X is in degree 0, too, applying $\mathrm{Hom}_{\hat{A}}(X,-)$ and $\mathrm{Hom}_A(X,-)$ to the $\hat{A}$-resolution and the A-resolution, respectively, yields isomorphic complexes.

$\Rightarrow \mathrm{Ext}^i_A(X,Y) \simeq \mathrm{Ext}^i_{\hat{A}}(X,Y) \ \forall i.$

By the definition of the shift as Ω^{-1}, $\mathrm{Ext}^{i+1}_{\hat{A}}(X,Y) \simeq \mathrm{Ext}^1_{\hat{A}}(X, Y[i])$ (dimension shift).

Thus it remains to identify $\mathrm{Ext}^1_{\hat{A}}(X,Z)$ with $\underline{\mathrm{Hom}}(X, Z[1])$ (for Z a shifted copy of Y).

There is an exact sequence $0 \to Z[1] \to I \to Z \to 0$ with I projective and injective.

Applying $\mathrm{Hom}_{\hat{A}}(X,-)$ shows that $\mathrm{Ext}^1_{\hat{A}}(X,Z)$ is isomorphic to the quotient of $\mathrm{Hom}_{\hat{A}}(X, Z[1])$ modulo maps factoring through I. $\square$

This should remind us of a property of derived categories discussed in chapter 13.

There is another property familiar from derived categories:

16.4 Theorem: The triangulated category $\hat{A}$-$\underline{\mathrm{mod}}$ has a t-structure $(\mathcal{M}^{\leq 0}, \mathcal{M}^{\geq 0})$ with heart equivalent to A-mod.

This theorem, due to Happel, follows from another theorem, also due to Happel:

16.5 Theorem: The embedding A-mod $\to \hat{A}$-$\underline{\mathrm{mod}}$ induces a fully faithful triangulated functor $F: D^b(A\text{-mod}) \to \hat{A}$-$\underline{\mathrm{mod}}$.

If $\mathrm{gl\,dim}(A) < \infty$, F is an equivalence.

This theorem shows that sometimes there are equivalences between derived and stable categories. This is not surprising from an abstract point of view. Categories of (co)chain complexes over additive categories always have a Frobenius exact structure and derived categories thus always can be viewed as stable categories of Frobenius exact categories. Generally, triangulated categories are called algebraic triangulated categories if they are stable categories of Frobenius exact categories. But it is rare to see such an explicit realisation as a stable category as in 16.5.

There are several proofs of 16.5 in the literature. Happel constructed the functor F by extending the functor $A\text{-mod} \to \hat{A}\text{-mod} \to \hat{A}\text{-mod}$ in 16.2 in several steps, which is rather technical. Another construction, due to Barot and Mendoza, is based on the following idea:

Since $A\text{-mod} \subset \hat{A}\text{-mod}$ by sending M to M_0, one can also send complexes of A-modules to $D^b(\hat{A}\text{-mod})$ and thus produces a functor $D^b(A\text{-mod}) \to D^b(\hat{A}\text{-mod})$.
What really needs to be done is to turn a complex in $\hat{A}\text{-mod}$ into an object of $\hat{A}\text{-mod}$, i.e. a module over $\hat{A}$.

Their idea is as follows: Given a complex $X^* = \dots \to X^{n-1} \to X^n \to X^{n+1} \to \dots$ one can modify it:

$$\begin{tikzcd}
 & \dots \arrow[r] & X^{n-1} \arrow[r, "d"] \arrow[d, equal] & X^n \arrow[r] \arrow[d, "\varepsilon"'] & X^{n+1} \arrow[r] \arrow[d] & X^{n+2} \arrow[r] \arrow[d, equal] & \dots \\
\text{to} & \dots \arrow[r] & X^{n-1} \arrow[r, "\varepsilon \circ d"] & I \arrow[r] & PO \arrow[r] & X^{n+2} \arrow[r] & \dots
\end{tikzcd}$$

where $X^n \xrightarrow{\varepsilon} I$ is an injective envelope, PO is the pushout, and all other terms of X are unchanged. Similarly one can modify

$$\begin{tikzcd}
 & \dots \arrow[r] & X^{n-1} \arrow[r, "d"] & X^n \arrow[r] & X^{n+1} \arrow[r] & X^{n+2} \arrow[r] & \dots \\
\text{to} & \dots \arrow[r] & X^{n-1} \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & P \arrow[r] \arrow[u, "p"'] & X^{n+2} \arrow[r] \arrow[u, equal] & \dots
\end{tikzcd}$$

where $P \xrightarrow{p} X^{n+1}$ is a projective cover and PB is the pullback.

When X^* is a bounded complex, finitely many steps of this kind allow to modify X^* into a complex $\dots \to I^{-2} \to I^{-1} \to X^0 \to P^0 \to P^1 \to \dots$ where the I^i are injective and the P^i are projective. (Since $\hat{A}$ is Frobenius, projective = injective.)

X° is an $\hat{A}$-module and the functor to be $D^b(\hat{A}\text{-mod}) \to \hat{A}\text{-}\underline{\text{mod}}$ should send X^* to X°.

In this way, an A-module X°, seen as an $\hat{A}$-complex, really is sent to itself.

This is not yet a functor and quite some work is needed to make it functorial. This includes checking that it turns quasi-isomorphisms into isomorphisms in $\hat{A}\text{-}\underline{\text{mod}}$.

To show that F is an equivalence of triangulated categories, when gldim $A<\infty$ Barot and Mendoza use a theorem of Rickard:

<u>16.6 Theorem:</u> Let A be a finite dimensional selfinjective algebra. Then there is an equivalence of triangulated categories

$$D^b(A\text{-mod})/K^b(A\text{-proj}) \simeq A\text{-}\underline{\text{mod}}$$

This gives another, different connection between derived and stable categories. It works not only for selfinjective algebras, but also for Frobenius algebras like $\hat{A}$, but it cannot work for algebras, where $A\text{-}\underline{\text{mod}}$ is not triangulated.

The proof is not long, but one needs to carefully define the quotient $D^b(A\text{-mod})/K^b(A\text{-proj})$ and check that it has a triangulated structure induced from that of $D^b(A\text{-mod})$.

When we are willing to believe these facts, we can check as follows that

$$\tilde{G}: A\text{-mod} \hookrightarrow D^b(A\text{-mod}) \twoheadrightarrow D^b(A\text{-mod})/K^b(A\text{-proj})$$

induces an equivalence of triangulated categories

$$G: A\text{-}\underline{\text{mod}} \xrightarrow{\sim} D^b(A\text{-mod})/K^b(A\text{-proj})$$

$\tilde{G}$ sends projective modules to zero, hence G exists. $\tilde{G}$ is full $\Rightarrow$ G is full.

If $G(f)=0$ for $f: X \to Y$, then f occurs in a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow$.

Then $G(f)=0$ implies $G(g)$ is split mono and $id_{G(Y)}$ factors through $G(g)$. Since G is full, the identity id_Y factors through g, which must be split mono. $\Rightarrow f=0 \Rightarrow G$ is faithful.

Finally we have to check that G is dense: An object in $D^b(A\text{-mod})/K^b(A\text{-proj})$ is represented by a complex of projectives: $P^* \to P^r \to P^{r+1} \to \dots \to P^s \to 0 \to \dots$

Since X^* has bounded cohomology we can assume that all cohomology occurs in degrees $r, \dots, s$. The morphism of complexes

$$\begin{tikzcd}
X^* \arrow[d, "\alpha"'] & \cdots \arrow[r] & P^{r-2} \arrow[r] \arrow[d, equal] & P^{r-1} \arrow[r] \arrow[d, equal] & P^r \arrow[r] \arrow[d, equal] & P^{r+1} \arrow[r] \arrow[d, "0"] & \cdots \arrow[r] & P^s \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\
Y^* & \cdots \arrow[r] & P^{r-2} \arrow[r] & P^{r-1} \arrow[r] & P^r \arrow[r] & 0 \arrow[r] & \cdots & 0 & &
\end{tikzcd}$$

has mapping cone in $K^b(A\text{-proj}) \Rightarrow \alpha$ is an isomorphism in $D^b(A\text{-mod})/K^b(A\text{-proj})$.
Y^* has cohomology only in degree r, hence it is isomorphic to a shifted module.

Now we return to the setting of 16.5. Applying 16.6 to $\hat{A}$, which is Frobenius, yields

$$D^b(\hat{A}\text{-mod})/K^b(\hat{A}\text{-proj}) \simeq \hat{A}\text{-}\underline{\text{mod}}$$

Therefore, one can replace $\hat{A}\text{-}\underline{\text{mod}}$ in 16.5 and consider instead the composition

$$D^b(A\text{-mod}) \longrightarrow D^b(\hat{A}\text{-mod}) \longrightarrow D^b(\hat{A}\text{-mod})/K^b(\hat{A}\text{-proj})$$

(One has to check that this is the composition of Rickard's equivalence with the functor in 16.5.)

Using the construction by Barot and Mendoza, this functor is shown to be full and faithful.

Now assume $\mathrm{gldim}(A) < \infty$. Then A-mod is a generating A-subcategory of $D^b(A\text{-mod})$. This means $D^b(A\text{-mod})$ is the smallest triangulated subcategory of $D^b(A\text{-mod})$ containing A-mod. And the assumption implies that A-mod also is a generating subcategory of $\hat{A}\text{-}\underline{\text{mod}}$. This implies the equivalence.

When $\mathrm{gldim}(A)$ is infinite, $D^b(A\text{-mod})$ and $\hat{A}\text{-}\underline{\text{mod}}$ really are different: $\hat{A}\text{-}\underline{\text{mod}}$ always has almost split triangles (the triangulated analogue of Auslander-Reiten sequences).
$D^b(A\text{-mod})$ however has almost split triangles if and only if $\mathrm{gldim}(A) < \infty$.