§ 15. t-structures

We return to the general setup of triangulated categories, which now are called $\mathcal{D}$ to avoid confusion with tilting complexes T.

15.1 Definition: Let $\mathcal{D}$ be a triangulated category. A pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full subcategories of $\mathcal{D}$ is called a t-structure on $\mathcal{D}$ if the following conditions are satisfied, using the notation $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$.

(I) $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$

(II) $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathrm{Ob}(\mathcal{D}^{\leq 0})$ and $Y \in \mathrm{Ob}(\mathcal{D}^{\geq 1})$

(III) For any object $X \in \mathrm{Ob}(\mathcal{D})$, there exists a distinguished triangle

$$X_0 \longrightarrow X \to X_1 \rightsquigarrow \text{ in } \mathcal{D} \text{ with } X_0 \in \mathrm{Ob}(\mathcal{D}^{\leq 0}) \text{ and } X_1 \in \mathrm{Ob}(\mathcal{D}^{\geq 1})$$

The full subcategory $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the heart of the t-structure. (This definition is due to Beilinson, Bernstein, Deligne and Gabber.)

When $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure, then $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$ is also a t-structure.

Let $\mathcal{D} = \mathcal{D}(\mathcal{A})$ a derived category of an abelian category, for instance of a module category. Let $\mathcal{D}^{\leq 0}(\mathcal{A})$ be the full subcategory with objects X satisfying $H^i(X) = 0$ for $i > 0$, and $\mathcal{D}^{\geq}(\mathcal{A})$ contain objects X with $H^i(X) = 0$ for $i < 0$. Then $(\mathcal{D}^{\leq 0}(\mathcal{A}), \mathcal{D}^{\geq 0}(\mathcal{A}))$ is a t-structure on $\mathcal{D}$: Given a complex

$$X^*: \ldots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to X^2 \to \ldots$$

$X^* \in \mathcal{D}^{\leq 0}(\mathcal{A})$ means Cohomology allowed $H^i(X) = 0$ for $i > 0$

$X^* \in \mathcal{D}^{\geq 0}(\mathcal{A})$ means $H^i(X) = 0$ for $i < 0$ cohomology allowed

$X^* \in \mathcal{D}^{\leq -1}(\mathcal{A})$ means cohomology allowed $H^i(X) = 0$ for $i > -1$ [1] is shift to the left
$= \mathcal{D}^{\leq 0}[1]$

$X^* \in \mathcal{D}^{\geq 1}(\mathcal{A})$ means $H^i(X) = 0$ for $i < 1$ Cohomology allowed [-1] is shift to the right
$= \mathcal{D}^{\geq 0}[-1]$

(This picture helps to remember the notation with $\leq$ and $\geq$)

Now we check the axioms in the cases $D^b(R\text{-Mod})$ and $D^-(R\text{-Mod})$:

(I) is shown by the above pictures.

(II) We may assume that $0 = X^1 = X^2 = \ldots = X^n = \ldots$ for $n > 0$, by truncating:

$$\begin{tikzcd}
\hat{X}^* \text{ truncated } X^*: & \cdots \arrow[r] & X^{-2} \arrow[r] \arrow[d, equal] & X^{-1} \arrow[r, "d^{-1}"] \arrow[d, equal] & \mathrm{Ker}(d^0) \arrow[r, "d^0"] \arrow[d, "\mathrm{incl}"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
X^*: & \cdots \arrow[r] & X^{-2} \arrow[r] \arrow[d] & X^{-1} \arrow[r, "d^{-1}"] \arrow[d] & X^0 \arrow[r, "d^0"] \arrow[d] & X^1 \arrow[r] & X^2 \arrow[r] & \cdots \\
\text{cone}: & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X^0/\mathrm{Ker}(d^0) \arrow[r] & X^1 \arrow[r] \arrow[u, equal] & X^2 \arrow[r] \arrow[u, equal] & \cdots
\end{tikzcd}$$

i.e. $\hat{X}^* = \tau_{\le 0}(X^*)$

is acyclic, isomorphic to a complex of injectives in $K^+(R\text{-Inj})$, which has null-homotopic identity $\Rightarrow$ this cone $\simeq 0$.

Since Y is exact in degrees ≤ 0, any map $\hat{X}^* \to Y$ is homotopic to zero, as $\hat{X}^*$ is isomorphic to a complex in $K^-(R\text{-Proj})$.

(III) The same truncation as in (II) helps: $0 \to \tau_{\le 0}(X^*) \to X^* \to X^*/\tau_{\le 0}(X^*) \to 0$ yields the distinguished triangle required.

$\Rightarrow$ $(D^{\le 0}, D^{\ge 0})$ is a t-structure on the derived category.

Here, $D^{\le 0} \cap D^{\ge 0}$ is equivalent to R-Mod: R-Mod is contained in $D^{\le 0} \cap D^{\ge 0}$.
Conversely, for $X^* \in D^{\le 0} \cap D^{\ge 0}$, truncation as above yields again $X^* \simeq \tau_{\le 0}(X^*)$, and identifying $\tau_{\le 0}(X^*)$ with a complex in $K^-(R\text{-Proj})$ implies $X^* \simeq H^0(X^*)$, which is an R-module.

$\Rightarrow$ We have found R-Mod as the heart of a t-structure on $D^b(R\text{-Mod})$.

t-structures are related to a familiar general concept:

15.2 Definition: Let $\mathcal{C}$ be an additive category and $\mathcal{T}$ and $\mathcal{F}$ two full subcategories. The pair $(\mathcal{T}, \mathcal{F})$ is called a *torsion theory* (or a *torsion pair*) on $\mathcal{C}$ $\Leftrightarrow$ the following conditions are satisfied:

(I) $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ for X in $\mathcal{T}$ and Y in $\mathcal{F}$.

(II) If $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ for all $Y \in \mathcal{F}$ then $X \in \mathcal{T}$.

(III) If $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ for all $X \in \mathcal{T}$ then $Y \in \mathcal{F}$.

Give an example of a torsion pair in the category $\mathcal{C} = \mathbb{Z}$-mod of finitely generated abelian groups.

15.3 Lemma: Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ is a torsion theory on $\mathcal{D}$.

Proof: (I) is part of the definition of t-structures.

(II) Let X in $\mathcal{D}$ with $\mathrm{Hom}_{\mathcal{D}}(X,Y) = 0$ for all Y in $\mathcal{D}^{\geq 1}$. The third property of a t-structure provides a distinguished triangle $\underset{\underset{\mathcal{D}^{\leq 0}}{\text{in}}}{X_0} \xrightarrow{f} X \xrightarrow{g} \underset{\underset{\mathcal{D}^{\geq 1}}{\text{in}}}{X_1} \rightsquigarrow$

By assumption on X, $g = 0$.

$\overset{12.9}{\Rightarrow}$ f is split epi and thus $X \in \mathcal{D}^{\leq 0}$.

(III) is proved dually. $\square$

For torsion pairs, objects in $\mathcal{T}$ often are called torsion objects and objects in $\mathcal{F}$ are called torsion free objects. This terminology is not used for t-structures. $\mathcal{D}^{\leq 0}$ often is called the aisle of the t-structure and $\mathcal{D}^{\geq 0}$ is called the co-aisle.

$\uparrow$ think of an aisle in a church, not in an airplane

15.3 may suggest to study $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$ instead of $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then, however, we would miss the following key property of t-structures:

15.4 Theorem: Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then the heart $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is an abelian category.

The proof needs a sequence of lemmas, to be stated and proved now; throughout we fix a triangulated category $\mathcal{D}$ and a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.

A t-structure provides an abstract setup to define truncation functors, which we so far only have seen in derived categories and in homotopy categories of complexes. (The "t" in t-structures seems to refer to the word truncation.)

For a given object X in $\mathcal{D}$, property $\boxed{iii}$ of the given t-structure provides us with a distinguished triangle $X_0 \to X \to X_1 \rightsquigarrow$ where X_0 is in $\mathcal{D}^{\le 0}$ and X_1 in $\mathcal{D}^{\ge 1}$.
We want to define $\tau_{\le 0} X := X_0$ and $\tau_{\ge 1} X := X_1$, and then $\tau_{\le n} X$ and $\tau^{\ge n} X$ by shifting appropriately.

We want to extend this definition on objects to the definition of functors $\tau_{\le 0}: \mathcal{D} \to \mathcal{D}^{\le 0}$ and $\tau_{\ge 1}: \mathcal{D} \to \mathcal{D}^{\ge 1}$ such that $\tau_{\le 0}$ is right adjoint to the inclusion $\mathcal{D}^{\le 0} \to \mathcal{D}$ and $\tau_{\ge 1}$ is left adjoint to the inclusion $\mathcal{D}^{\ge 1} \to \mathcal{D}$.

(Since we write $\mathcal{D}^{\le 0}$ and not $\mathcal{D}_{\le 0}$ we may also write $\tau^{\ge 0}$ instead of $\tau_{\le 0}$. Sometim such notation is used in the literature.)

The definition of functors and natural transformations needs uniqueness statements of the following kind:

15.5 Lemma: Let $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{h_1}{\rightsquigarrow} X[1]$ and $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{h_2}{\rightsquigarrow} X[1]$ be two distinguished triangles and suppose $\mathrm{Hom}_{\mathcal{D}}(X[1], Z) = 0$. Then $h_1 = h_2$.

Proof: In the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id"] & Y \arrow[r, "g"] \arrow[d, "id"] & Z \arrow[r, "h_1"] \arrow[d, dashed, "\exists \varphi"] & X[1] \arrow[d, "id"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h_2"] & X[1] \end{tikzcd}$$

there exists φ making the diagram commutative.

$\Rightarrow$ $g = \varphi \circ g$ and $h_1 = h_2 \circ \varphi$. In particular, $(id_Z - \varphi) \circ g = 0$. By the long exact sequence of the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(-, Z)$, $(id_Z - \varphi)$ factors through h_1:
$\exists\ \psi: X[1] \to Z$ such that $id_Z - \varphi = \psi \circ h_1$. By assumption, $\psi = 0$ $\Rightarrow$ $\varphi = id_Z$ $\Rightarrow h_1 = h_2$. $\square$

15.6 Proposition: (a) The inclusion $\mathcal{D}^{\le n} \longrightarrow \mathcal{D}$ has a right adjoint $\tau_{\le n}: \mathcal{D} \to \mathcal{D}^{\le n}$.
The inclusion $\mathcal{D}^{\ge n} \longrightarrow \mathcal{D}$ has a left adjoint $\tau_{\ge n}: \mathcal{D} \to \mathcal{D}^{\ge n}$.
(b) There exists a unique natural transformation $d: \tau_{\ge n+1} \to [1] \circ \tau_{\le n}$ such that for each object X in $\mathcal{D}$: $\tau_{\le n}(X) \to X \to \tau_{\ge n+1}(X) \overset{d}{\rightsquigarrow} \tau_{\le n}(X)[1]$ is a distinguished triangle.

Proof: The distinguished triangle $X_0 \to X \to X_1 \rightsquigarrow$ defines $\tau_{\le 0}(X)$.

We have to define $\tau_{\le 0}$ on morphisms $f: X \to Y$ in $\mathcal{D}$.
For Y there is a distinguished triangle $Y_0 \xrightarrow{\beta} Y \to Y_1 \rightsquigarrow$
Applying $\mathrm{Hom}_{\mathcal{D}}(X_0, -)$ yields a long exact sequence

$$\dashrightarrow \underset{\underset{0}{\shortparallel}}{\mathrm{Hom}_{\mathcal{D}}(X_0, Y_1[-1])} \to \mathrm{Hom}_{\mathcal{D}}(X_0, Y_0) \underset{\varphi}{\to} \mathrm{Hom}_{\mathcal{D}}(X_0, Y) \to \underset{\underset{0}{\shortparallel}}{\mathrm{Hom}_{\mathcal{D}}(X_0, Y_1)} \to$$

implying $\mathrm{Hom}_{\mathcal{D}}(X_0, Y_0) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(X_0, Y)$ is an isomorphism.

$\Rightarrow$ In

$$\begin{tikzcd}
X_0 \arrow[r, "\alpha"] \arrow[d, dashed, "\exists"'] & X \arrow[r] \arrow[d, "f"'] & X_1 \arrow[r, squiggly] & {} \\
Y_0 \arrow[r, "\beta"'] & Y \arrow[r] & Y_1 \arrow[r, squiggly] & {}
\end{tikzcd}$$

the composition $f\alpha$ factors through β, and the isomorphism φ (induced by β) gives a unique such factorisation, which we use as $\tau_{\le 0}(f)$ ^ by $\mathrm{Hom}_{\mathcal{D}}(X_0, Y_1[-1]) = 0$

$\rightsquigarrow$ $\tau_{\le 0}$ is now defined and it is a functor.

To check adjointness, let Z be any object in $\mathcal{D}_{\le 0}$. Applying $\mathrm{Hom}_{\mathcal{D}}(Z, -)$ to $X_0 \to X \to X_1 \rightsquigarrow$ yields the same kind of long exact sequence as above:

$$\dashrightarrow \mathrm{Hom}_{\mathcal{D}}(\underset{\underset{\mathcal{D}_{\le 0}}{\cap}}{Z}, \underset{\underset{0}{\shortparallel}}{X_1[-1])} \to \mathrm{Hom}_{\mathcal{D}}(Z, X_0) \to \mathrm{Hom}_{\mathcal{D}}(Z, X) \to \underset{\underset{0}{\shortparallel}}{\mathrm{Hom}_{\mathcal{D}}(Z, X_1)} \to$$

$$\begin{array}{lcl}
\text{implying } \mathrm{Hom}_{\mathcal{D}}(Z, X_0) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}}(Z, X) \\
\qquad\qquad \shortparallel & & \qquad \shortparallel \\
\qquad \mathrm{Hom}_{\mathcal{D}_0}(Z, X_0) & & \mathrm{Hom}_{\mathcal{D}}(\mathrm{incl}(Z), X) \\
\qquad\qquad \shortparallel & & \\
\qquad \mathrm{Hom}_{\mathcal{D}_0}(Z, \tau_{\le 0} X) & &
\end{array}$$

where incl is the inclusion
$\mathrm{incl}: \mathcal{D}_{\le 0} \to \mathcal{D}$

This is the formula needed to show that $\tau_{\le 0}$ is right adjoint to the inclusion. This proves (a) for $\tau_{\le 0}$.

The proof of (a) for $\tau_{\ge 1}$ is similar, and the same proofs work for the shifted categories $\mathcal{D}^{\le n}$ and $\mathcal{D}^{\ge n}$.

To prove (b) we apply Lemma 15.5 to get uniqueness of the map d. It remains to check that d defines a natural transformation: Let $f: X \to Y$ be any morphism in $\mathcal{D}$.

Then in

$$\begin{tikzcd}
\tau_{\le n} X \arrow[r] \arrow[d, "\tau_{\le n}(f)"] & X \arrow[r] \arrow[d, "f"'] & \tau_{\ge n+1} X \arrow[r, "d(X)"] \arrow[d, dashed, "\exists g"] & \tau_{\le n} X[1] \arrow[d, "\tau_{\le n}(f)[1]"] \\
\tau_{\le n} Y \arrow[r] & Y \arrow[r] & \tau_{\ge n+1} Y \arrow[r, "d(Y)"] & \tau_{\le n} Y[1]
\end{tikzcd}$$

g exists and makes the diagram commutative.

Since $\tau_{\le n} X[1] \in \mathcal{D}_{\le n}$ and $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_{\le n}, \mathcal{D}_{\ge n+1}) = 0$, commutativity of the right hand square implies uniqueness of g $\Rightarrow g = \tau_{\ge n+1}(f) \Rightarrow d$ is natural. $\square$

By definition of $\tau_{\le n}$ and $\tau_{\ge n}$, $\tau_{\le n}(X[m]) \simeq \tau_{\le n+m}(X)[m]$
and $\tau_{\ge n}(X[m]) \simeq \tau_{\ge n+m}(X)[m]$
for all X and all n, m.

15.7 Proposition: (a) For X in $\mathcal{D}^{\le n}$, $\tau_{\le n}(X) \to X$ is an isomorphism.
For X in $\mathcal{D}^{\ge n}$, $X \to \tau_{\ge n}(X)$ is an isomorphism.

(b) An object X in $\mathcal{D}$ is in $\mathcal{D}^{\le n} \Leftrightarrow \tau_{>n}(X) = 0$ (where $\tau_{>n} = \tau_{\ge n+1}$)
and it is in $\mathcal{D}^{\ge n} \Leftrightarrow \tau_{<n}(X) = 0$.

(c) Let $X' \to X \to X'' \rightsquigarrow$ be a distinguished triangle in $\mathcal{D}$. If X' and X'' are in $\mathcal{D}^{\ge 0}$, then so is X. If X' and X'' are in $\mathcal{D}^{\ge 0}$, then so is X. Similarly for $\mathcal{D}^{\le 0}$

check these statements for the standard example $D^b(R\text{-Mod})$ on pages 15.1 and 15.2

Proof: (a) follows from 15.6(b) and $\mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 1} = 0$.
(b) also follows from 15.6(b) and $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 1}) = 0$.
(c) for $\mathcal{D}^{\ge 0}$: Let X' and X'' be in $\mathcal{D}^{\ge 0}$. Then $\mathrm{Hom}_{\mathcal{D}}(\tau_{<0}X, X') = 0 = \mathrm{Hom}_{\mathcal{D}}(\tau_{<0}X, X'')$.
$\Rightarrow \mathrm{Hom}_{\mathcal{D}}(\tau_{<0}X, X) = 0 \Rightarrow \tau_{<0}X = 0$, and (b) can be applied. □

15.8 Corollary: Let $a, b \in \mathbb{Z}$.
(a) If $b \ge a$, then $\tau_{\ge b} \circ \tau_{\ge a} \simeq \tau_{\ge a} \circ \tau_{\ge b} \simeq \tau_{\ge b}$ and $\tau_{\le b} \circ \tau_{\le a} \simeq \tau_{\le a} \circ \tau_{\le b} \simeq \tau_{\le a}$
(b) If $b < a$, then $\tau_{\le b} \circ \tau_{\ge a} = \tau_{\ge a} \circ \tau_{\le b} = 0$

Proof: (a) 15.7(a) implies $\tau_{\ge a} \circ \tau_{\ge b} \simeq \tau_{\ge b}$ and $\tau_{\le a} \circ \tau_{\le b} \simeq \tau_{\le a}$.
Let X be in $\mathcal{D}$, Y in $\mathcal{D}^{\ge b}$. Then $\mathrm{Hom}_{\mathcal{D}^{\ge b}}(\tau_{\ge b}(\tau_{\ge a}X), Y) \underset{\text{by adjointness}}{=} \mathrm{Hom}_{\mathcal{D}}(\tau_{\ge a}X, Y) \underset{15.6}{\simeq}$
$\simeq \mathrm{Hom}_{\mathcal{D}}(X, Y) \underset{\text{by adjointness}}{\simeq} \mathrm{Hom}_{\mathcal{D}^{\ge b}}(\tau_{\ge b}X, Y) \Rightarrow \tau_{\ge b}(\tau_{\ge a}X) \simeq \tau_{\ge b}(X)$, and similarly the fourth isomorphism claimed.
(b) follows from 15.7(b). □

$\tau_{\le b} \circ \tau_{\ge a}$ is not in general 0, unless $b < a$, and similarly, $\tau_{\ge a} \circ \tau_{\le b}$ need not be 0.
However, the equation $\tau_{\le b} \circ \tau_{\ge a} = \tau_{\ge a} \circ \tau_{\le b}$ is a general fact:

15.9 Proposition: For all a, b: $\tau_{\geq a} \circ \tau_{\leq b} = \tau_{\leq b} \circ \tau_{\geq a}$.

More precisely, for any object X in $\mathcal{D}$ there exists a unique morphism $\varphi : (\tau_{\geq a} \circ \tau_{\leq b})(X) \longrightarrow (\tau_{\leq b} \circ \tau_{\geq a})(X)$ such that the diagram

$$\begin{tikzcd}
\tau_{\leq b}(X) \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geq a}(X) \\
(\tau_{\geq a} \circ \tau_{\leq b})(X) \arrow[rr, "\varphi"] & & (\tau_{\leq b} \circ \tau_{\geq a})(X) \arrow[u]
\end{tikzcd}$$

commutes.

This morphism φ is an isomorphism.

Check this for $D^b(R\text{-Mod})$ with the standard t-structure explained on pages 15.1 and 15.2.

Proof: By 15.8(b) we may assume $b \geq a$. By 15.8(a), $\tau_{\geq b} \circ \tau_{\geq a} \simeq \tau_{\geq b}$ etc and thus there are distinguished triangles

$$(*) \quad \tau_{\leq b} \tau_{\geq a} X \longrightarrow \tau_{\geq a} X \longrightarrow \tau_{> b} X \rightsquigarrow$$

$$\text{and } (**) \quad \tau_{< a} X \longrightarrow \tau_{\leq b} X \longrightarrow \tau_{\geq a} \tau_{\leq b} X \rightsquigarrow$$

} which are the distinguished triangles in 15.6(b)

Claim: Both $\tau_{\leq b} \tau_{\geq a} X$ and $\tau_{\geq a} \tau_{\leq b} X$ are in $\mathcal{D}^{\geq a} \cap \mathcal{D}^{\leq b}$.

We check it for $\tau_{\leq b} \tau_{\geq a} X$. This is in $\mathcal{D}^{\leq b}$, since $\tau_{\leq b}$ has image in this subcategory. Moreover, $\tau_{\geq a} X$ is in $\mathcal{D}^{\geq a}$ and by $b \geq a$ also $\tau_{> b} X$ is in $\mathcal{D}^{\geq a}$ $\overset{15.7(c)}{\Longrightarrow}$ the third term in the first triangle must be in $\mathcal{D}^{\geq a}$ as well. ✓

The morphism $\tau_{\leq b} X \longrightarrow X$ induces $(\tau_{\geq a} \circ \tau_{\leq b})(X) \xrightarrow{\alpha} \tau_{\geq a} X$. By the claim, $(\tau_{\geq a} \circ \tau_{\leq b})(X)$ does not map to $\tau_{> b} X$ $\Rightarrow$ α factors through $(\tau_{\leq b} \circ \tau_{\geq a})(X)$, which defines φ uniquely.

To show that φ is an isomorphism, we apply the octahedral axiom to

$$\begin{tikzcd}
& \tau_{\leq b} X \arrow[dr] & \\
\tau_{< a} X \arrow[ur] \arrow[rr] & & X
\end{tikzcd}$$

$$\begin{tikzcd}
& & \tau_{\geq a} \tau_{\leq b} X \arrow[ddr, "+"] & \\
& \tau_{\leq b} X \arrow[ur] \arrow[dr] & & \\
\tau_{< a} X \arrow[ur] \arrow[rr] & & X \arrow[r] \arrow[dr] & \tau_{\geq a} X \arrow[d, "\partial"] \\
& & & \tau_{> b} X \arrow[d, squiggly] \\
& & & {}
\end{tikzcd}$$

which yields the distinguished triangle

$$\tau_{\geq a} \tau_{\leq b} X \longrightarrow \tau_{\geq a} X \longrightarrow \underset{\tau_{> b} \tau_{\geq a} X}{\overset{\tau_{> b} X}{\shortparallel}} \rightsquigarrow$$

$$\Rightarrow \tau_{\geq a} \tau_{\leq b} X \simeq \tau_{\leq b} \tau_{\geq a} X$$

Comparing this with the triangle $(*)$ we get that φ is an isomorphism. $\square$

Why are we interested in these relations?
In our first example, the standard t-structure on $D^b(R\text{-Mod})$ we used cohomology H^i to define the aisle and the co-aisle. We will now see that we can do the same for any t-structure, provided we redefine H^i (a new H^i, not the old one) in terms of the t-structure. It is of course enough to define H^0 and then use shifts.

15.10 Definition: As before, let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ be the heart of this t-structure.
Define the functor $H^0 : \mathcal{D} \longrightarrow \mathcal{C}$ by

$$H^0(X) := (\tau_{\geq 0} \circ \tau_{\leq 0})(X) \simeq (\tau_{\leq 0} \circ \tau_{\geq 0})(X),$$

and define for $a \neq 0$: $H^a(X) = H^0(X[n]) \simeq (\tau_{\geq n} \circ \tau_{\leq n})(X)[n]$.

So, H^0 is cohomology relative to the given t-structure. It has values in the heart $\mathcal{C}$ of the t-structure, not in an abelian category used to define $\mathcal{D}$, for instance as derived category.

Check the example of the standard t-structure on $D^b(R\text{-Mod})$.

When X is in $\mathcal{D}^{\geq a}$ for some a, there is a distinguished triangle

$$\begin{tikzcd} H^a(X)[-a] \arrow[r] & \tau_{\geq a} X \arrow[r] \arrow[d, phantom, sloped, "\simeq"] & \tau_{>a} X \arrow[r, squiggly] & {} \\ & X & & \end{tikzcd}$$

why?

$\Rightarrow$ $X = \tau_{>a} X$ if and only if $H^a(X) = 0$.

Proof of Theorem 15.4:
The closure property in 15.7(c) implies for $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$:
If $X \to Y \to Z \rightsquigarrow$ is a distinguished triangle in $\mathcal{D}$ and X and Z are in $\mathcal{C}$, then Y is in $\mathcal{C}$ as well. So, $\mathcal{C}$ is kind of extension closed.
$\mathcal{D}$ is an additive category $\Rightarrow$ $\mathcal{C}$ is an additive category provided we can form (finite) direct sums: Let X_1 and X_2 be in $\mathcal{C}$ $\Rightarrow$ $X_1 \xrightarrow{\text{incl}} X_1 \oplus X_2 \xrightarrow{\text{proj}} X_2 \rightsquigarrow$ is a distinguished triangle $\Rightarrow$ $X_1 \oplus X_2$ is in $\mathcal{C}$.

We have to show that $\mathcal{C}$ has kernels and cokernels and that these are compatible.

Let $f: X \to Y$ be a morphism in $\mathcal{C}$. It occurs in a distinguished triangle $X \xrightarrow{f} Y \to Z \rightsquigarrow$ in $\mathcal{D}$. By 15.7 (applied to the triangle shifted), Z is in $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$.

Claim: $H^0(Z) \simeq \tau_{\geq 0} Z$ is a cokernel of f and $H^0(Z[-1]) \simeq \tau_{\leq 0}(Z[-1])$ is a kernel of f

Check that this works in the standard example.

To prove this claim, let W be any object in $\mathcal{C}$. Then there are two long exact sequences

— $\mathrm{Hom}_{\mathcal{D}}(X[1], W) \to \mathrm{Hom}_{\mathcal{D}}(Z, W) \to \mathrm{Hom}_{\mathcal{D}}(Y, W) \xrightarrow{f} \mathrm{Hom}_{\mathcal{D}}(X, W)$ and

— $\mathrm{Hom}_{\mathcal{D}}(W, Y[-1]) \to \mathrm{Hom}_{\mathcal{D}}(W, Z[-1]) \to \mathrm{Hom}_{\mathcal{D}}(W, X) \xrightarrow{f} \mathrm{Hom}_{\mathcal{D}}(W, Y)$ --

$W \in \mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0} \Rightarrow \mathrm{Hom}_{\mathcal{D}}(X[1], W) = 0 = \mathrm{Hom}_{\mathcal{D}}(W, Y[-1])$.

Also, $\mathrm{Hom}_{\mathcal{D}}(Z, W) \simeq \mathrm{Hom}_{\mathcal{D}}(\tau_{\geq 0} Z, W) \Rightarrow 0 \to \mathrm{Hom}_{\mathcal{D}}(\tau_{\geq 0} Z, W) \to \mathrm{Hom}_{\mathcal{D}}(Y, W) \to \mathrm{Hom}_{\mathcal{D}}(X, W)$ is exact, hence $\tau_{\geq 0} Z \in \mathcal{C}$ is the cokernel of f.

Similarly, $\mathrm{Hom}_{\mathcal{D}}(W, Z[-1]) = \mathrm{Hom}_{\mathcal{D}}(W, \tau_{\leq 0} Z[-1]) \Rightarrow \tau_{\leq 0}(Z[-1])$ is the kernel of f.

(We use that $\mathcal{C}$ is a full subcategory, hence $\mathrm{Hom}_{\mathcal{D}} = \mathrm{Hom}_{\mathcal{C}}$ whenever it makes sense.)

This shows the existence of kernels and of cokernels in $\mathcal{C}$.

Finally we have to verify that the canonical map coker(ker) $\to$ ker(coker) is an isomorphism. More down to earth this means the two ways to define the image of a morphism do coincide:

$$\begin{tikzcd}
 & X \arrow[rrr, "f"] \arrow[dr] & & & Y \arrow[dr] & \\
\ker f \arrow[ur] & & X/\ker f \arrow[r, phantom, "\overset{!}{\simeq}"] & \ker(\mathrm{coker}) \arrow[ur] & & \mathrm{coker} f \\
 & & \simeq \mathrm{coker}(\ker) & & &
\end{tikzcd}$$

In our situation, $\mathrm{coker} f \simeq \tau_{\geq 0} Z$.

There is a distinguished triangle $I \to Y \to \tau_{\geq 0} Z \rightsquigarrow$ By 15.7(c), I is in $\mathcal{D}^{\geq 0}$.

Again we use the octahedral axiom:

$$\begin{tikzcd}
 & & X[1] \arrow[ddr, bend left] & \\
 & Z \arrow[ur] \arrow[dr] & & \\
Y \arrow[ur] \arrow[rr] & & \tau_{\geq 0} Z \arrow[r] \arrow[d, bend right] & I[1] \arrow[dl] \\
 & & \tau_{<0} Z[1] \arrow[dl, squiggly] \arrow[d, phantom, "\shortparallel"] & \\
 & {} & \tau_{\leq 0} Z &
\end{tikzcd}$$

The resulting distinguished triangle implies that is in $\mathcal{D}^{\leq 0}$ $\Rightarrow I$ is an object of $\mathcal{C}$.

Now we will identify I with the two ways of defining the image:

15.1

$\ker f \simeq \tau_{\leq 0}(Z[-1])$ occurs in a distinguished triangle

$$\ker f \to X \to I \rightsquigarrow, \text{ hence } I \simeq \operatorname{Coker}(\ker f)$$

$\operatorname{Coker} f \simeq \tau_{\geq 0} Z$ occurs in a distinguished triangle

$$I \to Y \to \operatorname{Coker} f, \text{ hence } I \simeq \ker(\operatorname{Coker} f)$$

(and the isomorphism is the canonical one, since the maps in these triangles are the canonical ones). $\square$

At this point one can also show:

If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an exact sequence in the abelian category $\mathcal{C}$, then there exists a unique morphism $h: Z \to X[1]$ in $\mathcal{D}$ such that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \overset{h}{\rightsquigarrow} X[1]$$

is a distinguished triangle in $\mathcal{D}$.

When $\mathcal{D}$ is a derived module category, the heart $\mathcal{C}$ of a t-structure need not be a module category. Example: $D^b(k(\circ \rightrightarrows \circ)) \simeq D^b(\operatorname{coh}(\mathbb{P}^1(k)) \Rightarrow \exists$ t-structure on $D^b(A\text{-mod})$ with $\mathcal{C} \simeq \operatorname{coh}(\mathbb{P}^1(k))$.

Kronecker algebra A

$\uparrow$ coherent sheaves over the projective line

This $\mathcal{C}$ has enough injective objects, but no non-zero projectives. Positively seen, there are derived equivalences relating algebra and geomet.

Moreover, in general $D^b(\mathcal{C})$ ($\mathcal{C}$ any heart of a t-structure) need not be equivalent to $\mathcal{D}$. For instance, if $\mathcal{C}$ is a module category, its progenerators need not be tilting complexes in $\mathcal{D}$. They may have negative self-extensions in $\mathcal{D}$ (ie they may be silting complexes).