Happel's theorem 14.4 was extended several times in several ways. The first such extension, due to Cline, Parshall and Scott, works for general rings, not just finite dimensional algebras, and drops the assumption of finite global dimension.

14.5 Definition: Let A be a ring and T a left A-module. T is called a tilting module :⟺ (T1) T has a finite resolution by finitely generated projective A-modules,

(T2) $\mathrm{Ext}_A^n(T,T)=0 \quad \forall n>0$

(T3) $_AA$ has a finite resolution $0\to A\to T^0\to T^1\to \dots \to T^d\to 0$ where all $T^i\in \mathrm{add}\,T$.

↑ direct summands of finite direct sums of T

The concept of tilting module has been discovered in representation theory of finite dimensional algebras, by Brenner and Butler, Bongartz, Happel and Ringel.
The precise definition depends on the context. A (classical) tilting module of a finite dimensional algebra A is finite dimensional, satisfies $\mathrm{pdim}\,T\le 1$ (⇒(T1)), as well as (T2) and (T3) for $d\le 1$.
A progenerator is a tilting module. check
Because of (T1) and of (T3), injective modules may fail to be tilting modules.

14.6 Theorem (Cline, Parshall and Scott): Let A be a ring, $_AT$ a tilting module and $B=\mathrm{End}_A(T)^{op}$. Then $D^b(A\text{-}Mod)\simeq D^b(B\text{-}Mod)$, i.e. A and B are derived equivalent.

Their proof follows similar lines as the proof of Morita's theorem given in chapter 9.
Since T is an A-B-bimodule, there are functors $A\text{-}Mod \underset{T\otimes_B -}{\overset{\mathrm{Hom}_A(T,-)}{\rightleftarrows}} B\text{-}Mod$.
By general theory known to them (but not yet to us),
these and their left/right derived functors can be turned into functors

$$D^b(A\text{-}Mod) \underset{T\overset{L}{\otimes}_B -}{\overset{R\mathrm{Hom}_A(T,-)}{\rightleftarrows}} D^b(B\text{-}Mod)$$

14.11

These two functors are an adjoint pair, and under the assumptions of 14.6, they are mutually quasi-inverse equivalences.
Cline, Parshall and Scott also have shown that T_B then is a tilting module over B and the situation is symmetric in A and B. Moreover, $\operatorname{pdim}(T_B) \le d$.
When A and B are finite dimensional algebras, then left or right exact functor between A-Mod and B-Mod inducing derived equivalences must be $\operatorname{Hom}(T,-)$ or $T \otimes -$ for T a tilting module.
Cline, Parshall and Scott did, however, not show that all derived equivalences are of this form, and as we will see this is not true, in a non-trivial way.

Now we consider some examples of tilting modules, which by 14.4 or 14.6 induce derived equivalences.
Let $A = k(\circ \to \circ)$, i.e. type A_2. Then A-ind is

$$\begin{matrix} & k\to k & \\ \nearrow & & \searrow \\ 0\to k & & k \to 0 \end{matrix}$$

${}_AA$ is a tilting module, but also $T = \begin{matrix} k\to k =: T_1 \\ \oplus \\ k \to 0 =: T_2 \end{matrix}$
Check this: $k \to k$ is projective
$k \to 0$ is injective, its minimal projective resolution is $0 \to \begin{smallmatrix}0\\ \downarrow \\ k\end{smallmatrix} \to \begin{smallmatrix}k\\ \downarrow \\ k\end{smallmatrix} \to \begin{smallmatrix}k\\ \downarrow \\ 0\end{smallmatrix} \to 0 \quad (*)$
$\Rightarrow$ (T1) ✓
For (T2) we only have to check $\operatorname{Ext}^n_A(k\to 0, k \to k) = 0$ for $n=1$ which follows from $(*)$ or from an Auslander–Reiten formula.
For (T3) we have to coresolve the indecomposable projective modules (as required in 14.4). $0 \to \begin{smallmatrix}k\\ \downarrow \\ k\end{smallmatrix} \to \begin{smallmatrix}k\\ \downarrow \\ k\end{smallmatrix} \to 0$ ✓ For $0 \to k$, $(*)$ works.

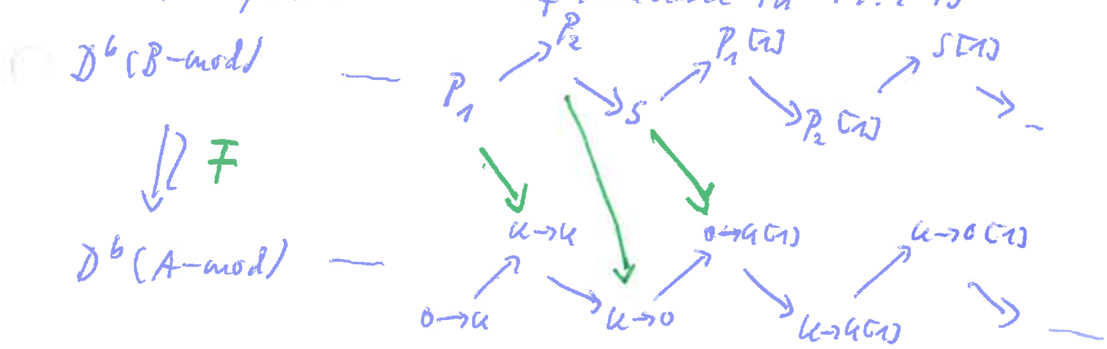
$B = \operatorname{End}_A(T)^{op}$ is isomorphic to A itself. To describe Happel's equivalence we need to fix the additive equivalence $B\text{-proj} \simeq \operatorname{add} T$
Since $\operatorname{Hom}_A(T_1, T_2) = k$ and $\quad P_1 \longmapsto T_1$
$\operatorname{Hom}_A(T_2, T_1) = 0$, P_1 must be the $\quad P_2 \longmapsto T_2$
simple projective B-module and P_2 the other indecomposable projective B-module.
The derived equivalence $D^b(B\text{-mod}) \xrightarrow{\sim} D^b(A\text{-mod})$ by construction sends P_1 to T_1 and P_2 to T_2. What happens to $\underbrace{P_2/P_1}_{=:S}$, the simple injective B-module?

In B-mod there is an exact sequence $0 \to P_1 \to P_2 \to S \to 0 \Rightarrow$ In $K^b(B\text{-proj})$ there is a distinguished triangle $P_1 \to P_2 \to S \rightsquigarrow P_1[1]$. In $K^b(\operatorname{add} T)$ this becomes a distinguished triangle $T_1 \to T_2 \to X \rightsquigarrow T_1[1]$ and we have to determine X. The map $T_1 \to T_2$ occurs in the short exact sequence (in A-mod)

$$(*) \quad 0 \to \begin{matrix} 0 \\ \downarrow \\ k \end{matrix} \to \begin{matrix} k \\ \downarrow \\ k \end{matrix} \to \begin{matrix} k \\ \downarrow \\ 0 \end{matrix} \to 0, \qquad \begin{matrix} k \\ \downarrow \\ k \end{matrix} = T_1, \quad \begin{matrix} k \\ \downarrow \\ 0 \end{matrix} = T_2$$

and in the resulting distinguished triangle $\Rightarrow X \simeq \begin{matrix} 0 \\ \downarrow \\ k \end{matrix}[1]$

$\rightsquigarrow$ The picture for the equivalence in 14.4 is



Classical tilting theory is older than 14.4 and did not use derived categories. Instead it compared certain pieces of A-mod and B-mod. In the example this comparison would just say $\operatorname{add}(B\text{-proj}) \simeq \operatorname{add}(T)$ and $\operatorname{add}(S) \simeq \operatorname{add}(0 \to k)$. In the picture of the AR quiver this means $\operatorname{add} S$ (on the right hand side of $P_1 \nearrow P_2 \searrow S$) is equivalent to $\operatorname{add}(0 \to k)$ (on the left hand side of $0\to k \nearrow k\to k \searrow k \to 0$). This vaguely explains the term "tilting" which indicates a kind of coordinate change.

If you have seen a proof of Gabriel's theorem on finite representation type of quivers, this example may, and should, remind you of reflection functors, which in fact fit into the setup of classical tilting.

To get a class of examples let A be a finite dimensional algebra which has a simple projective module S (for $A = kQ/I$, S belongs to a sink in Q), and suppose A is elementary. Moreover, assume S is not injective. Thus there exists an almost split sequence $0 \to S \to X \to \tau^{-1} S \to 0$. (*)

Claim: all direct summands of X are projective.
Proof: Assume X has a non-projective indecomposable direct summand X_0. Then X_0 is the end term of an almost split sequence $0 \to \tau(X_0) \to Y \to X_0 \to 0$. (**)
By (*) there is an irreducible map $S \to X_0$. $\Rightarrow$ S is a direct summand of Y (up to isomorphism) and there is an irreducible map $\tau(X_0) \xrightarrow{f} S$. But S is simple projective $\Rightarrow$ f splits ↯.
$\Rightarrow$ The almost split sequence (*) has projective middle term P

$$(*)\quad 0 \to S \to \underbrace{\oplus P_i}_{P} \to \tau^{-1}S \to 0 \quad (P_i \text{ indecomposable projective with } S/\mathrm{rad}\, P_i)$$

Claim: $T := \tau^{-1}(S) \oplus P$ is a tilting module.
Proof: (*) implies (T1) and (T3).
(T2) follows from $D\,\mathrm{Hom}_A(T, \underset{S}{\tau T}) = 0$.

Such tilting modules are called APR-tilting modules, since they have been discovered and used by Auslander, Platzeck and Reiten.
If you are familiar with reflection functors, then you may observe that $\mathrm{Hom}_A(T,-)$ on ind $A - \{S\}$ behaves like the reflection functor associated with a sink.
While this functor sends S to 0, the derived equivalence associated with T sends it to a shifted B-module.

The derived equivalence in 14.4 sends B to the tilting module $M = T$. The shift, which is an autoequivalence of $D^b(A\text{-mod})$, sends A to $A[1]$, which is not a module. Hence this equivalence cannot come from a tilting module. In general, there is no reason for a derived equivalence to send the regular module to a module.
This is actually good news. For some algebras, there are tilting modules except the progenerators: Let A be self-injective and T a tilting module. By (T3), ${}_AA$ is contained in $T^0 \in \mathrm{add}\,T$. ${}_AA$ injective implies $A \mid T^0$, hence ${}_AA \in \mathrm{add}\,T$. A general property we have not yet checked states that the number of non-isomorphic indecomposables in add T equals that in add A $\Rightarrow$ T is projective. (If it wouldn't, it would be a counterexample to the Auslander-Reiten conjecture, see chapter 7.)

14.7 Definition: Let R be a ring. A <u>tilting complex</u> over R satisfies the following conditions:

- T is an object in $K^b(R\text{-proj})$ (a finite complex of finitely generated projectives)
- $\mathrm{Hom}_{D^b(R)}(T, T[n]) = 0$ $\forall n \in \mathbb{Z} - \{0\}$ (no positive or negative self-extensions)
- The category $\mathrm{add}\, T \subset K^b(R\text{-proj})$ generates $K^b(R\text{-proj})$ as a triangulated category (there is no proper triangulated subcategory of $K^b(R\text{-proj})$ containing all of $\mathrm{add}\, T$)

These conditions still are in the spirit of classical Morita theory.

14.8 Theorem (Rickard): Let A and B be rings. Then the following assertions are equivalent.

(a) $D^b(A\text{-Mod})$ and $D^b(B\text{-Mod})$ are equivalent as triangulated categories.

(b) $K^-(A\text{-Proj})$ and $K^-(B\text{-Proj})$ are equivalent as triangulated categories.

(c) $K^b(A\text{-Proj})$ and $K^b(B\text{-Proj})$ are equivalent as triangulated categories.

(d) $K^b(A\text{-proj})$ and $K^b(B\text{-proj})$ are equivalent as triangulated categories.

(e) There exists a tilting complex T over R with endomorphism ring $\mathrm{End}_{D^b(R)}(T)^{op} = S$.

There are further equivalent conditions, about equivalences on $D(A\text{-Mod})$-level or on D^-, D^+, etc. For finite dimensional algebras one also can replace A-Mod by the finite dimensional modules A-mod.

When A has finite k-dimension, B has so, too, since the tilting complex T over A has only finitely many non-zero terms each of which is finite dimensional.

This theorem often is refered to as Rickard's derived Morita theory.

There is, however a major difference to classical Morita theory: T is not a complex of S-modules: S is not the endomorphism ring of the (chain or cochain) complex, it is the endomorphism ring up to homotopy. Under mild conditions that are always satisfied for finite dimensional algebras, Rickard has shown the existence of two-sided tilting complexes, whose terms are, for instance, $A \otimes_k B^{op}$-bimodules.

A different approach, due to Keller, in the more general context of differential graded algebras (i.e. the algebras are complexes themselves) produces two-sided tilting objects directly and then uses Hom- or tensor functors to prove a derived Morita theorem, under the mild assumptions needed for two-sided tilting complexes in Rickard's approach.

Rickard's proof construct the derived equivalenc in a similar way as Happel did it, but using K^- and therefore infinite complexes, which makes the proof much longer and more complicated.

A satisfies the properties in 14.7. check! ⇒ condition (e) in 14.8. follows from each of the other conditions.

The derived equivalence gets constructed for large categories, e.g. $K^-(\text{Proj})$, assuming (e). And then Richard show that such an equivalence restricts to one on K^b or D^b level. Actually he proves more:

<u>14.9 Proposition:</u> Let R and S be rings. Then:

(a) Any triangle equivalence (= equivalence of triangulated categories) $K^-(R\text{-Proj}) \xrightarrow{\sim} K^-(S\text{-Proj})$ restricts to a triangle equivalence $D^b(R\text{-Mod}) \xrightarrow{\sim} D^b(S\text{-Mod})$.

(b) Any triangle equivalence $D^b(R\text{-Mod}) \xrightarrow{\sim} D^b(S\text{-Mod})$ restricts to a triangle equivalence $K^b(R\text{-Proj}) \xrightarrow{\sim} K^b(S\text{-Proj})$.

(Formally, these categories are not contained in each other, but we may identify $D^b(R\text{-Mod})$ with the full subcategory of $K^-(R\text{-Proj})$, consisting of objects with bounded homology, and similarly for $K^b(R\text{-Proj})$.)

Sometimes the properties in (a) and (b) are formulated as $D^b(R\text{-Mod})$ being a characteristic subcategory of $K^-(R\text{-Proj})$ etc (compare: a characteristic subgroup H of a group G is mapped to itself under each automorphism of G).

Proof: We need intrinsic conditions for an object X in $K^-(R\text{-Proj})$ to belong or not to $D^b(R\text{-Mod})$ etc, ie a condition in terms of X and $K^-(R\text{-Proj})$ only.

(a) Let X and Y be objects of $K^-(R\text{-Proj})$. If X has bounded (co)homology, then $\mathrm{Hom}_{K^-}(Y, X[i]) = 0$ when X is shifted so far to the right that Y cannot see the terms of X where there is (co)homology, ie for i large (in cochain notation). Conversely, when X has unbounded (co)homology, then $\mathrm{Hom}_{K^-}(R, X[i]) \neq 0$ for arbitrarily large i, since $\mathrm{Hom}_{K^-}(R,-)$ sees the (co)homology.

$\Rightarrow$ X has bounded (co)homology $\Leftrightarrow$ $\forall Y\ \exists N: \mathrm{Hom}_{K^-}(Y, X[n]) = 0\ \forall n \geq N$

X is in $D^b(R\text{-Mod})$

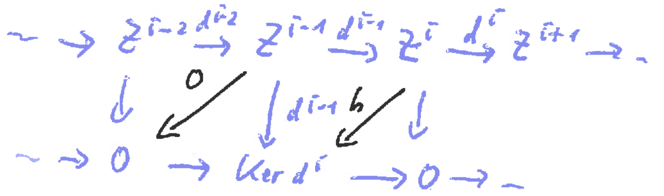
$$\begin{array}{ccccccc} & 0 & \to & R & \to & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & X^{i-1} & \xrightarrow{d^{i-1}} & X^i & \xrightarrow{d^i} & X^{i+1} & \to \cdots \end{array}$$

where $R \not\downarrow \mathrm{Im}(d^{i-1}) \subsetneq \mathrm{Ker}(d^i)$

(b) Now let X and Y be objects in $D^b(R\text{-Mod})$. When X is isomorphic to a bounded complex of projectives, then $\mathrm{Hom}_{K^-}(X, Y[i]) = 0$ for Y shifted to the left so far that X cannot see the terms of $Y[i]$ with (co)homology (and thus any morphism is null-homotopic, since X has projective terms only).

In any case, X is isomorphic to an object Z in $K^-(R\text{-Proj})$, and when X is not isomorphic to a bounded complex of projectives, for small i the kernel of $d^i: Z^i \to Z^{i+1}$ is not always projective (otherwise trivial direct summands of Z can be split off to turn Z into a bounded complex). Let Y be the direct sum of all these non-projective kernels, considered as a module = a complex concentrated in degree 0.

$\Rightarrow$ $\mathrm{Hom}_{K^-}(X, Y[i]) = \mathrm{Hom}_{K^-}(Z, Y[i]) \neq 0$ for such i except possibly the finitely many where (co)homology occurs:



Assume there is no (co)homology, ie $\mathrm{Ker}\, d^i = \mathrm{Im}\, d^{i-1}$. If there would be a homotopy $h: Z^i \to \mathrm{Ker}\, d^i$, then the isomorphism $Z^{i-1}/\mathrm{Ker}\, d^{i-1} \simeq \mathrm{Im}\, d^{i-1} = \mathrm{Ker}\, d^i$ would factor through h, ie through the projective module Z^i $\Rightarrow$ $\mathrm{Ker}\, d^i \mid Z^i$, but $\mathrm{Ker}\, d^i$ cannot be a direct summand of a projective module $\lightning$.

Hence, bounded complexes X of projectives in $D^b(R\text{-Mod})$ are characterised by $\mathrm{Hom}_{K^-}(X, Y[i]) = 0$ for all Y, i small. $\square$

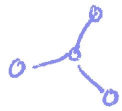
We still have to see an example of a tilting complex and a derived equivalence not induced by a tilting module, for instance one between self-injective algebras.

Let A be the a k-algebra given by the quiver $\overset{1}{\circ} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \overset{2}{\circ} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \overset{3}{\circ}$ modulo relations $\alpha_1\alpha_2 = 0$, $\beta_2\beta_1 = 0$, $\beta_1\alpha_1 = \alpha_2\beta_2$, all paths of length 3 vanish.

Then ${}_AA$ has Loewy series:
$$\underset{A(1)}{\begin{array}{c}1\\2\\1\\\shortparallel\end{array}} \oplus \underset{A(2)}{\begin{array}{c}2\\1\ \ 3\\2\\\shortparallel\end{array}} \oplus \underset{A(3)}{\begin{array}{c}3\\2\\3\\\shortparallel\end{array}}$$

Let B be the Nakayama algebra given by the quiver  and the relations that all paths of length 4 vanish.

Then ${}_BB$ has Loewy series:
$$\underset{B(1)}{\begin{array}{c}1\\2\\3\\1\\\shortparallel\end{array}} \oplus \underset{B(2)}{\begin{array}{c}2\\3\\1\\2\\\shortparallel\end{array}} \oplus \underset{B(3)}{\begin{array}{c}3\\1\\2\\3\\\shortparallel\end{array}}$$

Both A and B are self-injective. If you are familiar with Brauer tree algebras:
A has Brauer tree o—o—o—o and B has Brauer tree . Blocks (indecomposable algebra summands) of group algebras of finite groups are Brauer tree algebras when they have finite representation type and are not simple.

We define a complex T in $K^b(A\text{-proj})$ by

$$T := \begin{array}{l} \sim 0 \to \begin{smallmatrix}1\\2\\1\end{smallmatrix} \to 0 \sim \\ \quad\oplus \\ \sim 0 \to \begin{smallmatrix}1\\2\\1\end{smallmatrix} \to \begin{smallmatrix}2\\1\ 3\\2\end{smallmatrix} \to 0 \sim \\ \quad\oplus \\ \sim 0 \to \begin{smallmatrix}1\\2\\1\end{smallmatrix} \to \begin{smallmatrix}2\\1\ 3\\2\end{smallmatrix} \to \begin{smallmatrix}3\\2\\3\end{smallmatrix} \to 0 \end{array}$$

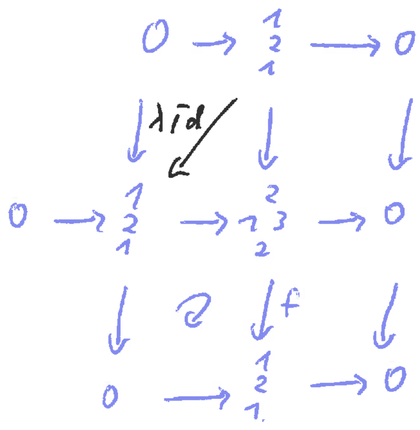
where the maps between indecomposable projectives are always non-zero (and then unique up to scalar).

This is a cochain complex in $K^b(A\text{-proj})$.

$\Rightarrow$ The first condition in 14.7 is satisfied.

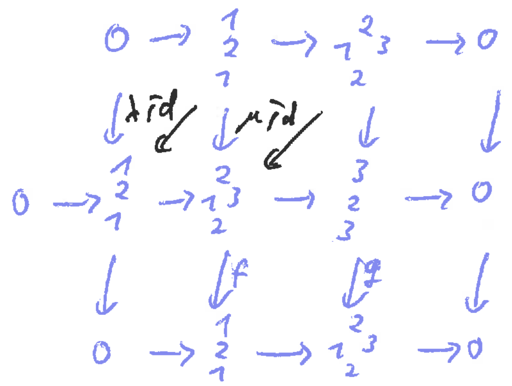
We check the other conditions in 14.7:

$\mathrm{Hom}_{K^b}(T, T[n]) = 0$ for $n \neq 0$: Many Hom's vanish since T and $T[n]$ don't see each other.



is homotopic to zero, for a scalar λ

commutativity of the left hand square forces f to vanish



is homotopic to zero, for scalars λ, μ

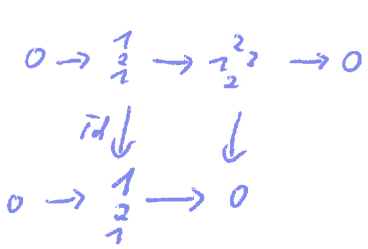
Commutativity of the squares forces f and g to be zero

and similarly for the other cases. check the details

$\Rightarrow$ In this way the second condition is verified.

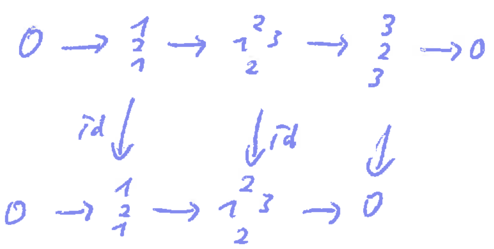
The third condition requires T to be a generator. We can verify this condition by "generating" $A(1)$, $A(2)$ and $A(3)$ in terms of distinguished triangles as follows: $A(1)$ is a direct summand of T. ✓

The morphism



extends to a distinguished triangle with cone a (shifted) copy of $A(2)$. ✓

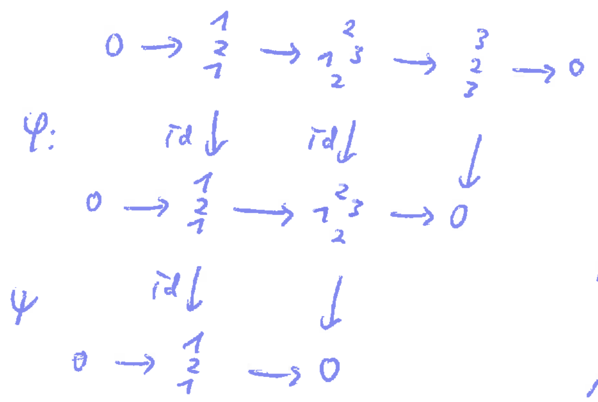
The morphism



extends to a distinguished triangle whose cone is – up to shift – $A(3)$. ✓

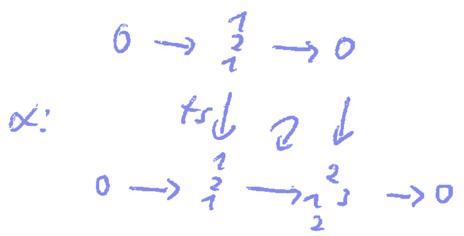
$\Rightarrow$ T is a tilting complex.

⇒ There is a derived equivalence between A and the endomorphism ring $\mathrm{End}_{D^b}(T)^{op}$, which we still have to determine:



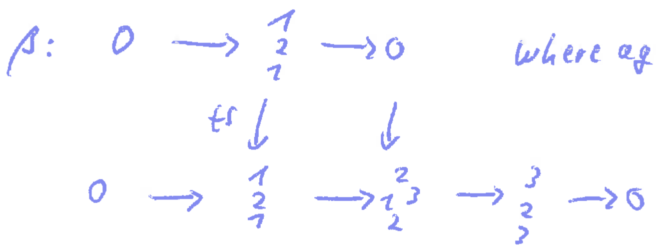
These are non-zero morphisms, since homotopies to zero would force some indecomposable projectives to be direct summands of others.
And they are unique up to sclar multiple.

But there are further morphisms between the summands of T:



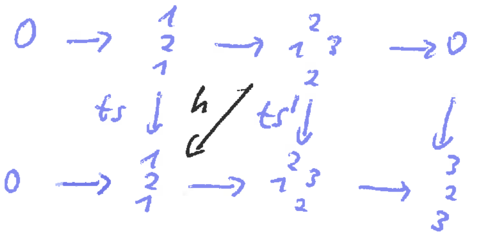
where ts: $\begin{smallmatrix}1\\2\\1\end{smallmatrix} \to \begin{smallmatrix}1\\2\\1\end{smallmatrix}$ sends the top to the socle

and similarly



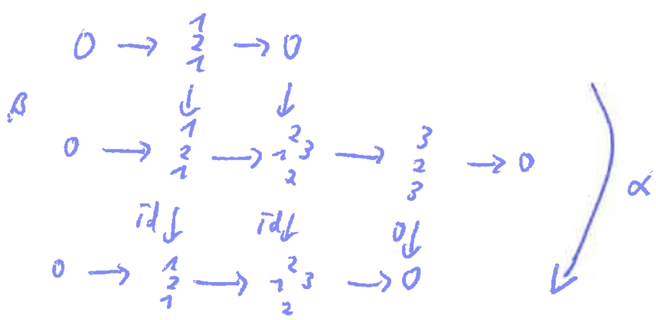
where again ts sends top to socle

and



where each of ts and ts' are sending top to socle with independent scalars, but using a homotopy h we may, for instance, assume $ts' = 0$

α factors through β:



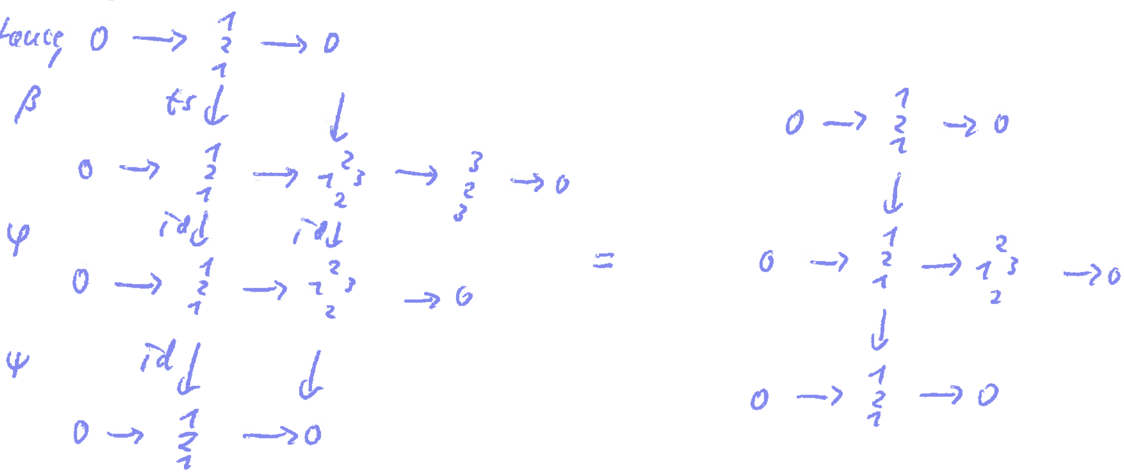
and γ also factors through α. in which way?

⇒ $\mathrm{End}_{D^b}(T)^{op}$ is generated by φ, ψ and β.

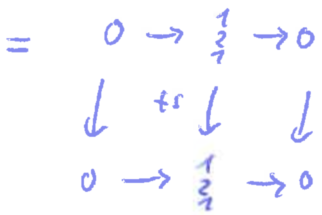
This tells us already the quiver of $\mathrm{End}_{\mathcal{D}^b}(T)^{op}$:  where the arrows are given by φ, ψ and β.

Paths of length three do not vanish, because of the identity maps in φ and ψ.

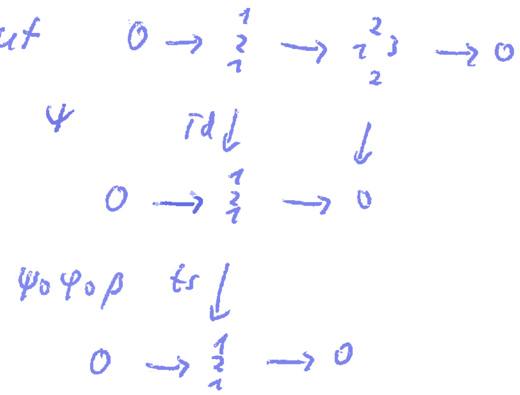
For instance

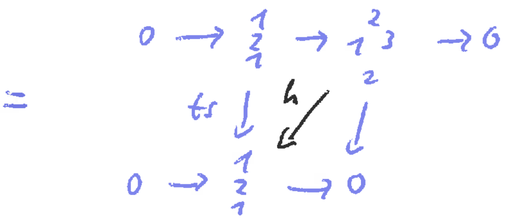


which has no chance to be homotopic to zero.



But





which is homotopic to zero

check the other cases as well

Result: $\mathrm{End}_{D^b(A)}(T)^{op} \simeq B$, the above Nakayama algebra.

$\Rightarrow$ A and B are derived equivalent.

Using this kind of tilting complexes, Rickard classified Brauer tree algebras up to derived equivalence.

Moreover, an algebra derived equivalent to a Brauer tree algebra must be a Brauer tree algebra itself – when we define the class of Brauer tree algebras to be closed under Morita equivalences.

Blocks of group algebras are Brauer tree algebras when their defect group (a block analogue of a Sylow subgroup) is cyclic. A major open conjecture, Broué's conjecture, claims the existence of certain derived equivalences a block with abelian defect and a block of another, smaller group algebra. Rickard's above derived equivalences such as the one in our example, settle (and have motivated) this conjecture in the case of cyclic defect.

Comparing classical Morita theory with Rickard's derived Morita theory 14.8, the latter is lacking a uniqueness statement that in the classical case is implied by the theorem of Eilenberg and Watts.
Of course, a derived category can have many auto-equivalences. The question is, however: Is a derived equivalence $D^b(A) \xrightarrow{\sim} D^b(B)$ uniquely determined by choosing a tilting complex T to be mapped to B? Rickard has shown that two such equivalences F and G must satisfy $F(X) \simeq G(X)$ for each object. But is an open problem to prove or disprove that F and G are naturally isomorphic as functors.