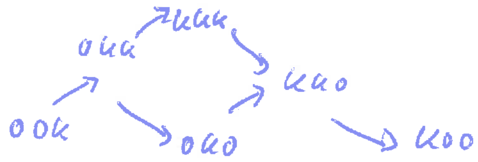
§ 14. Derived equivalences

We start by looking at algebras associated with type A_3: o—o—o. k is any field.

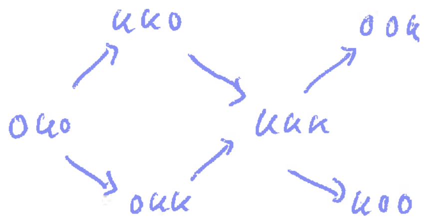
$k(\cdot \to \cdot \to \cdot) \simeq \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix}$, ARQ:



A

k-dimension 6
hereditary

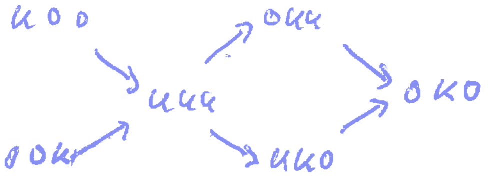
$k(\cdot \to \cdot \leftarrow \cdot) \simeq \begin{pmatrix} k & 0 & 0 \\ k & k & k \\ 0 & 0 & k \end{pmatrix}$, ARQ:



B

k-dimension 5
hereditary

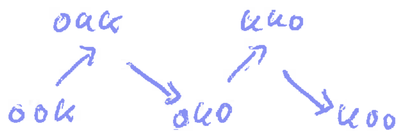
$k(\cdot \leftarrow \cdot \to \cdot) \simeq \begin{pmatrix} k & k & 0 \\ 0 & k & 0 \\ 0 & k & k \end{pmatrix}$, ARQ:



C

k-dimension 5
hereditary

$k(\cdot \to \cdot \to \cdot) \simeq \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}$



D

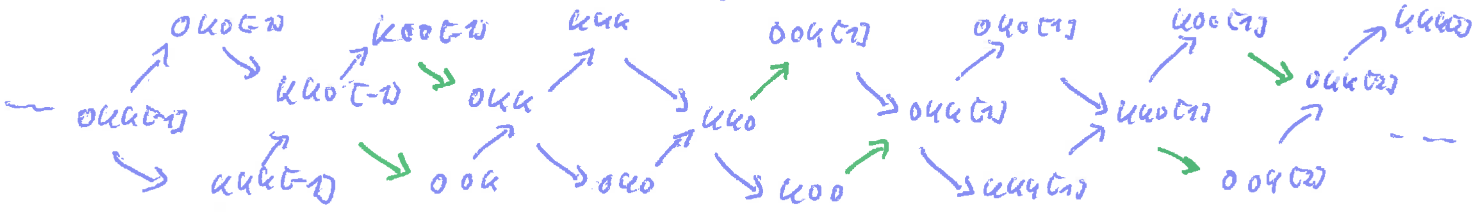
k-dimension 5
gldim = 2

There are various ways to see that these four algebras A, B, C and D represent four different Morita equivalent classes; for instance:

- All four algebras, hence Morita equivalence means isomorphism, and no two of them are isomorphic.
- Morita equivalence preserves the properties simple + projective, simple + injective, projective + injective, global dimension
  simple + projective ↔ sink in Q; simple + injective ↔ source in Q; global dimension ↔ $\mathrm{Ext}^n \neq 0, n = \mathrm{gldim}$
- Morita equivalence preserves the Auslander-Reiten quiver. why?

Still, there are some common properties; for instance, A, B, C each has 6 indecomposable modules, up to isomorphisms.

A, B and C are hereditary, hence indecomposable objects in the derived category are shifted modules. For $A = k(\bullet \to \bullet \to \bullet)$ we get:



This is the Auslander–Reiten quiver of $D^b(A\text{-mod})$, but we have not shown that this gadget exists. Anyway, by Theorem 13.2, the indecomposable objects, up to isomorphism, are those in the picture (when extended to an infinite picture).
The blue arrows are in $A\text{-mod} \subset D^b(A\text{-mod})$ representing the irreducible maps in the Auslander–Reiten quiver of A-mod. And the other blue arrows are just shifted copies of these. All of these arrows occur in exact sequences of modules, appropriately shifted. For instance,

$$\begin{tikzcd}[column sep=small] & 0kk \arrow[dr] & \\ 00k \arrow[ur] & & 0k0 \end{tikzcd}$$

is an almost split sequence in A-mod.

$$\begin{tikzcd}[column sep=small] 00k[1] \arrow[dr] & & 0k0[1] \\ & 0kk[1] \arrow[ur] & \end{tikzcd}$$

is just a shifted copy of that.

The green arrows also come (up to shift) from exact sequences of modules, by considering the induced triangles. For instance,

$$\begin{tikzcd}[column sep=small] & kkk \arrow[dr] & \\ 00k \arrow[ur] & & kk0 \end{tikzcd}$$

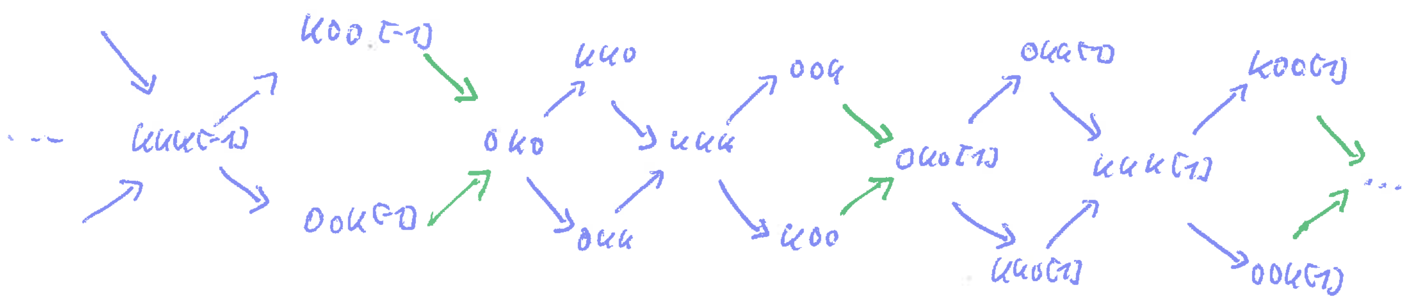
is an exact sequence. Hence there must be a distinguished triangle

$$\begin{tikzcd}[column sep=small] & kkk \arrow[dr] & \\ 00k \arrow[ur] & & kk0 \arrow[ll, "{[1]}"] \end{tikzcd}$$

with a morphism $kk0 \longrightarrow 00k[1]$.

(There are more such arrows. Doing some book keeping one can check that the green ones are irreducible, so the picture represents all irreducible morphisms in $D^b(A)$.)

For $D^b(B\text{-mod})$, there is an analogous picture

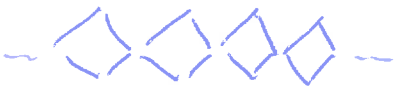


Here, green arrows are indistinguished triangles

$$0k0 \to kk0 \to k00 \rightsquigarrow 0k0[1]$$

and $0k0 \to 0kk \to 00k \rightsquigarrow 0k0[1]$

Schematically, both $D^b(A\text{-mod})$ and $D^b(B\text{-mod})$ come with the picture (of indecomposables)



and it looks like $\underset{D^b(A\text{-mod})}{D^b(A)}$ and $\underset{D^b(B\text{-mod})}{D^b(B)}$ are equivalent, as triangulated categories.

Not surprisingly, $D^b(C)$ looks the same:

$$\begin{tikzcd}[column sep=small]
 & k00 \arrow[dr] & & 0kk \arrow[dr] & & 00k[1] \arrow[dr] & \\
\sim & & kkk \arrow[ur] \arrow[dr] & & 0k0 \arrow[ur] \arrow[dr] & & kkk[1] \quad \sim \\
 & 00k \arrow[ur] & & kk0 \arrow[ur] & & k00[1] \arrow[ur] &
\end{tikzcd}$$

Where do the green arrows come from?

So, by experimental evidence, A, B and C have equivalent derived categories.

What about the fourth algebra, D? It has global dimension 2, hence there are indecomposable complexes whose cohomology is not concentrated in one degree only. $00k$, $kk0$ and $0kk$ are projective and $k00$ is is indecomposable of projective dimension two: $0 \to 00k \to 0kk \to kk0 \to k00 \to 0$ is the minimal projective resolution of $k00$. $0 \to 0kk \to kk0 \to 0$ in $K^b(D\text{-proj})$ has cohomology in two degrees:

cohomology: $00k \quad k00$

So we have found a sixth indecomposable object. And we could check by hand that there is no other indecomposable object in $K^b(D\text{-proj}) = D^b(D\text{-mod})$. This does however not imply $D^b(A\text{-mod}) \simeq D^b(D\text{-mod})$. We don't even know where to put the sixth object, the complex $X = \begin{matrix} 0kk \\ \downarrow \\ kk0 \end{matrix} \in K^b(D\text{-proj})$ in the picture of $D^b(A\text{-mod})$

It may be tempting to put X at the place of kkk (which is not D-module).
However, life isn't always easy.

Putting X in between the projective modules $0kk$ and $kk0$ to fill the gap requires the existence of (irreducible) morphisms $0kk \to X \to kk0$. We still can choose in which degree X lives. But:

$$\left.\begin{tikzcd} 0kk \arrow[r] \arrow[d] & 0kk \arrow[d] \\ 0 \arrow[r] & kk0 \end{tikzcd}\right\} = X$$

doesn't work $\Rightarrow$

the only chance is
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0kk \arrow[d] \\ 0kk \arrow[r] & kk0 \end{tikzcd}$$
But then
$$\begin{tikzcd} 0kk \arrow[r] \arrow[d] & 0 \arrow[d] \\ kk0 \arrow[r] & kk0 \end{tikzcd}$$
doesn't work.

The only reasonable maps are

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0kk \arrow[r, "Id"] \arrow[d] & 0kk \arrow[d] \\ kk0 \arrow[r, "Id"] & kk0 \arrow[r] & 0 \end{tikzcd}$$

These look irreducible, but they go in the wrong direction.

$\Rightarrow$ The situation must be more complicated, for some reason (or $D^b(A) \not\cong D^b(D)$). In fact, there is no reason, why an irreducible map between modules should remain irreducible in the derived category – it could factor through objects that are not (shifted) modules. And this happens here:

Consider the irreducible map between modules $kk0 \overset{\varphi}{\twoheadrightarrow} k00$, or the corresponding map between projective resolutions

$$\varphi: \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 00k \arrow[d] \\ 0 \arrow[r] \arrow[d] & 0kk \arrow[d] \\ kk0 \arrow[r] & kk0 \end{tikzcd}$$

In $D^b(D\text{-mod}) = K^b(D\text{-proj})$, φ factors through the complex X:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 00k \arrow[d] \\ 0 \arrow[r] \arrow[d] & 0kk \arrow[r] \arrow[d] & 0kk \arrow[d] \\ kk0 \arrow[r] & kk0 \arrow[r] & kk0 \end{tikzcd}$$

(and this factorisation is very natural). $\Rightarrow$ In the picture, X must be somewhere between the module $kk0$ (on the left of X) and the module $k00$ (on the right of X).

$\Rightarrow$ The AR-picture for $D^b(D\text{-mod})$ is rather unclear. Instead of experimenting and guessing we better start working on a construction of derived equivalences, which will clarify the situation in the above examples.

The following construction, due to Happel, is the first construction of derived equivalences between finite dimensional algebras. It was the first step towards a "derived Morita theory", which subsequently has been developed by Rickard and in different way by Keller. We carry out Happel's approach in detail and afterwards discuss more general results.
We restrict to finite dimensional k-algebras A and B, which have finite global dimension. We also restrict to bounded derived categories of finite dimensional modules.
Recall Morita equivalence: A and B are Morita equivalent $\Leftrightarrow \exists$ A-progenerator P with endomorphism ring (over A) isomorphic to $B \Leftrightarrow \exists$ B-progenerator Q with endomorphism ring (over B) isomorphic to A. The equivalence $A\text{-Mod} \to B\text{-Mod}$ then sends P to B. The image under the equivalence of an A-module M is determined by choosing a resolution of M by projectives, ie by objects in $\mathrm{add}\,P$ and using that the equivalence is exact and sends P to B.
Idea: Use projectivisation to generalise this: Let A be an algebra and $_AM$ a left A-module. Then Yoneda's Lemma implies an equivalence of additive categories

$$\mathrm{Hom}_A(M,-): \mathrm{add}\,M \xrightarrow{\sim} \mathrm{proj}\,E, \text{ where } E = \mathrm{End}_A(M)$$

(See the problem sheet on Gabriel quivers and Auslander-Reiten quivers, where M is a right A-module.)
Thus, for any M we can try to define a functor

$$(*) \quad K^b(E\text{-proj}) \longrightarrow K^b(A\text{-mod})$$

by sending E to $M \in A\text{-mod}$
a finite complex of E-projectives to a finite complexes with terms in $\mathrm{add}\,M$
If we only consider E of finite global dimension, then each E-module is quasi-isomorphic to a complex in $K^b(E\text{-proj})$ and the equivalence in 12.12 restricts to an equivalence of triangulated categories $K^b(E\text{-proj}) \simeq D^b(E\text{-mod})$.
The localisation functor $K^b(A\text{-mod}) \longrightarrow D^b(A\text{-mod})$ then allows to define $D^b(E\text{-mod}) \simeq K^b(E\text{-proj}) \xrightarrow{(*)} K^b(A\text{-mod}) \to D^b(A\text{-mod})$, provided we find a functor $(*)$. If A also has finite global dimension, we also can use the equivalence $K^b(A\text{-proj}) \simeq D^b(A\text{-mod})$.

So, let A be any finite dimensional k-algebra and M an A-module. Since $\operatorname{add} M \subset A\text{-mod} \subset K^b(A\text{-mod})$, we can define a functor, which is the

Composition $F : K^b(\operatorname{add} M) \xrightarrow[\text{inclusion}]{} K^b(A\text{-mod}) \xrightarrow[\text{localisation}]{} D^b(A\text{-mod})$

Because of $\operatorname{add} M \simeq E\text{-proj}$,

F is more or less the functor we want. We have to find criteria for F to be an equivalence, ie full and faithful and dense.

14.1 Lemma: Assume M has no self-extensions, ie $\operatorname{Ext}_A^i(M, M) = 0 \ \forall i > 0$. Then F is full and faithful.

Proof: First let $M_1, M_2 \in \operatorname{add} M$. Then $\operatorname{Hom}_A(M_1, M_2) = \operatorname{Hom}_{D^b(A\text{-mod})}(M_1, M_2)$ as we have checked in Chapter 13. Similarly, $\operatorname{Hom}_A(M_1[n], M_2[n]) = \operatorname{Hom}_{D^b(A)}(M_1[n], M_2[n])$ for all $n \in \mathbb{Z}$, while $\operatorname{Hom}_{D^b(A)}(M_1[p], M_2[q]) = 0$ for any $p \neq q$, by assumption.

Since there are no non-zero morphisms $\cdots 0 \to M_1 \to 0 \to \cdots$ between

complexes $\cdots \cdots 0 \to 0 \to 0 \to \cdots \to M_2 \to \cdots$

where M_1 and M_2 appear in different degrees, F is fully faithful on $\operatorname{add} M$ and all $\operatorname{add} M[n]$, $n \in \mathbb{Z}$.

Now we continue by induction on the width of complexes in $K^b(\operatorname{add} M)$, where $\cdots \to 0 \to 0 \to \underset{\neq 0}{M_1} \to \cdots \to \underset{\neq 0}{M_n} \to 0 \to 0 \to \cdots$ has width n. The case width $= 1$ has been settled.

A complex $X^\bullet$ of width > 1, say $\to 0 \to X^{-n} \to \cdots \to X^{-1} \to X^0 \to 0 \to \cdots$ can be truncated and fits into a distinguished triangle

$$(\dagger)\quad \begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & X^{-n} & & X^{-n} \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & X^{-1} & & X^{-1} \\ \downarrow & & \downarrow & & \downarrow \\ X^0 & & X^0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \\ \text{truncated} & & X^\bullet & & \text{truncated} \end{array}$$

For any $Y^\bullet$, there are two cohomological functors $\operatorname{Hom}_{K^b(\operatorname{add} M)}(Y^\bullet, -)$ and $\operatorname{Hom}_{D^b(A)}(Y^\bullet, -)$. Applying them to $(\dagger)$ and using the long exact cohomology sequence and double induction (also on the width of $Y^\bullet$) the isomorphism $\operatorname{Hom}_{K^b(\operatorname{add} M)}(Y^\bullet, X^\bullet) \simeq \operatorname{Hom}_{D^b(A)}(Y^\bullet, X^\bullet)$ follows. $\square$

A projective module M satisfies the criterion in 14.1. This is good, since we want Morita equivalences to be particular examples of derived equivalences.
When F is fully faithful, $\mathrm{Hom}_{K^b(\mathrm{add}\,M)}(M, M[n])$ must be isomorphic to $\mathrm{Ext}^n_A(M, M)$ for all $n \in \mathbb{Z}$. For $n > 0$ this forces the vanishing of self-extensions of M.
From examples we know that there many modules without self-extensions, for instance all indecomposable representations of Dynkin quivers.

A projective module needs to be a generator (and finitely generated) to induce a Morita equivalence. Similarly, M has to be "large enough" to get F dense.
P progenerator implies (when gldim $A < \infty$) that every A-module has a finite resolution with terms in add P.
Now, asking every A-module to have an add M-resolution would be too restrictive: A resolution $0 \to M_1 \to \dots \to M_n \to A \to 0$ forces $A \in \mathrm{add}\,M$. A better idea is to use coresolutions of A: $0 \to A \to M_1 \to \dots \to M_n \to 0$ (and just for A, not for other modules such as the injectives). For $M = A$, $0 \to A \to M = A \to 0$ is such a resolution, for P progenerator the same resolution works because of $A \in \mathrm{add}\,P$.
And the complex $0 \to M_1 \to \dots \to M_n \to 0$ has cohomology concentrated in one degree, and

$$\begin{array}{ccccccccccc} \dots \to & 0 & \to & A & \to & 0 & \to & \dots & \to 0 & \to & 0 \\ & & & \downarrow & & \downarrow & & & \downarrow & & \\ \dots \to & 0 & \to & M_1 & \to & M_2 & \to & \dots & \to M_n & \to & 0 \end{array}$$

is a quasi-isomorphism $\Rightarrow$ A is in the image of F inside $D^b(A\text{-mod})$.

Since we assume finite global dimension, there is an upper bound on the length of such an M-coresolution:

<u>14.2 Lemma:</u> Assume that M has no self-extensions, pdim $M \leq r$ and A has a finite M-coresolution. Then A has a finite M-coresolution of length $s \leq r$:

$$0 \to {}_AA \to M^0 \to M^1 \to \dots \to M^s \to 0$$

Proof: By assumption, there is a finite M-coresolution of A and we can choose one of minimal length $0 \to A \to M^0 \to M^1 \to \dots \to M^{s-1} \xrightarrow{f} M^s \to 0$.
Assume $s > r$. Let $X := \mathrm{Ker}(f)$.

Splitting the $\mathcal{M}$-coresolution of A into short exact sequences

$$0 \to A \to M^0 \to Y_0 \to 0$$
$$0 \to Y_0 \to M^1 \to Y_1 \to 0$$
$$\vdots$$
$$0 \to Y_{s-3} \to M^{s-2} \to Y_{s-2} \to 0 \quad \text{with } Y_{s-2} = X$$
$$0 \to X \to M^{s-1} \xrightarrow{f} M^s \to 0$$

and applying $\mathrm{Hom}_A(M,-)$ to each ses, producing a long exact cohomology sequence shows: $\mathrm{Ext}^1_A(M,X) \simeq \mathrm{Ext}^2_A(M,Y_{s-3}) \simeq \mathrm{Ext}^3_A(M,Y_{s-4}) = \ldots$ because all $\mathrm{Ext}^i_A(M, \mathrm{add}\, M)$-terms vanish.
Moreover, $\mathrm{pdim}\, M \leq r < s$ implies $\mathrm{Ext}^{>r}_A(M,-) = 0$, hence $\mathrm{Ext}^1_A(M,X) = 0$.
$\Rightarrow$ $0 \to X \to M^{s-1} \to M^s \to 0$ is split exact, which contradicts the minimality of s. ▯

Using the same splitting of the $\mathcal{M}$-resolution of A and then applying $\mathrm{Hom}(-,M)$ yields a long exact sequence $0 \to E^s \to \ldots \to E^1 \to E^0 \to M \to 0$ of right E-modules, with all $E^i \in \mathrm{add}\, E$. In other words, this is a projective resolution of M_E. (And it is finite without using $\mathrm{gldim}\, E < \infty$.)
For $e = e^2 \in A$, $M_E = eM \oplus (1-e)M$ as right E-module. $\Rightarrow$ eM has a projective E-resolution that is finite, too, and that corresponds to a finite $\mathcal{M}$-coresolution of the projective A-module Ae, if that exists.

<u>14.3 Lemma:</u> Suppose M has no self-extensions and each indecomposable projective A-module has a finite $\mathcal{M}$-coresolution. Then F is dense.
Proof: By assumption, each projective A-module has a finite $\mathcal{M}$-coresolution and thus it is quasi-isomorphic to a complex in $K^b(\mathrm{add}\, M)$. Since F is a triangulated functor, its image is triangulated and therefore contains all finite complexes of projective A-modules. As $\mathrm{gldim}\, A < \infty$, $D^b(A\text{-mod}) \simeq K^b(A\text{-proj})$
$\Rightarrow$ F is dense. ◻

Now we have arrived at Happel's theorem, which is a special case of the Morita theorem for derived module categories.

14.4 Theorem: Let A and B be finite dimensional k-algebras of finite global dimension. Let M be a finite dimensional A-module such that

- $\mathrm{Ext}_A^i(M,M) = 0 \ \forall i > 0$ and,
- each indecomposable projective A-module has a finite M-coresolution,
- and $\mathrm{End}_A(M)^{op} \cong B$.

Then the triangulated categories $D^b(A\text{-mod})$ and $D^b(B\text{-mod})$ are equivalent.

Proof: The assumptions allow to apply 14.1 and 14.4. $\Rightarrow$ F is dense.
Since the additive categories $\mathrm{add}\, M$ and $\mathrm{add}\, B$ are equivalent, their homotopy categories are equivalent as well. $\Rightarrow K^b(\mathrm{add}\, M) \simeq K^b(B\text{-proj})$.
Because of $\mathrm{gldim}\, B < \infty$, $D^b(B\text{-mod}) \simeq K^b(B\text{-proj})$.
This yields the equivalence of triangulated categories:

$$\begin{tikzcd}
D^b(B\text{-mod}) \arrow[rrr, "\sim"] \arrow[dr, "\sim"'] & & & D^b(A\text{-mod}) \\
& K^b(B\text{-proj}) \arrow[r, "\simeq"] & K^b(\mathrm{add}\, M) \arrow[r] & K^b(A\text{-mod}) \arrow[u]
\end{tikzcd}$$

$\square$

What happens to B?

$$\begin{tikzcd}
B \arrow[dr, mapsto] & & & M \\
& B \arrow[r, mapsto] & M \arrow[r, mapsto] & M \arrow[u, mapsto]
\end{tikzcd}$$

What happens when M is a progenerator?