§13. Modules as objects of derived categories

Let R be a ring and R-Mod its module category. Other categories associated with R are homotopy categories $K(R\text{-Mod})$, $K^-(R\text{-Proj})$, $D(R\text{-Mod})$, $D^b(R\text{-Mod})$, $D^-(R\text{-Mod})$, etc.

R-Modules X and Y are objects of R-Mod, but also of $K(R\text{-Mod})$, $D^b(R\text{-Mod})$, $D(R\text{-Mod})$, —, but not in general of $K^-(R\text{-Proj})$ or $K^+(R\text{-Inj})$. In the latter categories, X and Y are kind of represented by their projective or injective resolutions, respectively.

Moreover, in all the triangulated categories there also shifted versions $X[n]$, $Y[n]$ for all $n \in \mathbb{Z}$.

So, there are many questions arising:

- Is $\mathrm{Hom}(X,Y)$ the same in all these categories, which contain X and Y?
- Is $X \simeq X[n]$ for some or all $n \in \mathbb{Z}$?
- Is $\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}(P_*(X), P_*(Y))$ when $P_*(X)$ and $P_*(Y)$ are projective resolutions of X and Y, respectively?
- Does $\mathrm{Hom}(X, Y[n])$ have a meaning or is it just a formal datum?
- Are all objects in a derived category just sums of shifted modules?
- Can we see $\mathrm{Ext}^n_R(X,Y)$ somewhere in the derived category?
- How does one do computations in the derived or homotopy categories?
- Was it worth to define these complicated categories?

In this chapter we start addressing these questions, focussing on computations.

The situation is:

$$\begin{tikzcd}
\underset{X,Y}{R\text{-Mod}} \arrow[rr] \arrow[dr] & & K(R\text{-Mod}) \arrow[rr, "\text{localisation}"] & & D(R\text{-Mod}) \\
& Ch(R\text{-Mod}) \arrow[ur] & & & \\
& & \underset{P_*(X),\, P_*(Y)}{K^-(R\text{-Proj})} \arrow[uu, "\substack{\text{full sub-}\\ \text{category}}"'] \arrow[rr, "\text{equivalence}", "12.17"'] & & D^-(R\text{-Mod}) \arrow[uu, "\text{full subcategory}"']
\end{tikzcd}$$

R-Mod is a full subcategory of $Ch(R\text{-Mod})$, $X \mapsto \ldots \to 0 \to 0 \to X \to 0 \to 0 \to \ldots$

$Y \mapsto \ldots \to 0 \to 0 \to Y \to 0 \to 0 \to \ldots$

and there are no homotopies between modules $\Rightarrow$ $R\text{-Mod} \longrightarrow K(R\text{-Mod})$ is fully faithful.

Since $[n]$ is an auto-equivalence for each $n \in \mathbb{Z}$, $R\text{-Mod} \to K(R\text{-Mod})$, $X \mapsto X[n]$ (n fixed) is fully faithful as well.

$\Rightarrow$ There are many copies of R-Mod inside $K(R\text{-Mod})$ or $D(R\text{-Mod})$: each $R\text{-Mod}[n]$, $n \in \mathbb{Z}$.

X and $P_*(X)$ are both in $K(R\text{-Mod})$ (already in $K^-(R\text{-Mod})$) and in $D(R\text{-Mod})$.

In $K^-(R\text{-Mod})$ they are in general not isomorphic:

$$\begin{array}{ccccccccccc} P_*(X) & \cdots \to & P_2 & \to & P_1 & \to & P_0 & \to 0 \to 0 \to & \cdots \\ \downarrow\varphi & & \downarrow & & \downarrow & & \downarrow & & \\ X & \cdots \to & 0 & \to & 0 & \to & X & \to 0 \to 0 \to & \cdots \end{array}$$

is a quasi-isomorphism, but there may be no map backwards, not even a module homomorphism $X \to P_0$ that is not 0.

In $D(R\text{-Mod})$ (and already in $D^b(R\text{-Mod})$) this qis φ becomes invertible and $X \simeq P_*(X)$. This implies:

$\underset{\text{in general}}{\mathrm{Hom}(X,Y) \not\simeq} \mathrm{Hom}(\underset{X \simeq}{X}, P_*(Y))$ in $K^-(R\text{-Mod})$ why?

but $\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}(P_*(X), P_*(Y))$ in $D^-(R\text{-Mod})$ (and in $D^b(R\text{-Mod})$, $D(R\text{-Mod})$)

$\qquad\qquad \simeq \mathrm{Hom}(P_*(X), P_*(Y))$ in $K^-(R\text{-Proj})$

(Thus we better use $\mathrm{Hom}_{R\text{-Mod}}$, $\mathrm{Hom}_{K^-(R\text{-Mod})}$ etc in order to avoid confusion.)

So, $\mathrm{Hom}_{R\text{-Mod}}(X,Y) = \mathrm{Hom}_{K^-(R\text{-Proj})}(X,Y) = \mathrm{Hom}_{D^-(R\text{-Mod})}(X,Y)$ (same additive group and also same composition of morphisms).

$\Rightarrow$ We have a choice where to do computations. $K^-(R\text{-Proj})$ often is a better choice than the equivalent category $D^-(R\text{-Mod})$.

Let $P \neq 0$ be projective and consider, in $K^-(R\text{-Proj})$, the objects

$$P = P[0] \quad \cdots \to 0 \to 0 \to P \to 0 \to 0 \to \cdots$$

$$P[1] \quad \cdots \to 0 \to P \to 0 \to 0 \to 0 \to \cdots$$

$$P[-1] \quad \cdots \to 0 \to 0 \to 0 \to P \to 0 \to \cdots$$

There are no non-zero maps between any two of these complexes, apart from the endomorphisms.

Now let X and Y be any non-zero modules and choose projective resolutions

$P^*(X)$ $\ldots \to P^3 \to P^2 \to P^1 \to P^0 \to 0 \to 0 \to \ldots$ of X, and

$P^*(Y)$ $\ldots \to Q^3 \to Q^2 \to Q^1 \to Q^0 \to 0 \to 0 \to \ldots$ of Y, and their shifts

$P^*(X)[1]$ $\ldots \to P^2 \to P^1 \to P^0 \to 0 \to 0 \to 0 \to \ldots$ etc.

Can there be a morphism $f: P^*(X) \to P^*(Y)[-1]$?

$$
\begin{tikzcd}
P^*(X) \arrow[d, "f"] & \ldots \arrow[r] \arrow[dl, dashed, "\exists h_3"'] & P^3 \arrow[r] \arrow[d, "f_3"] \arrow[dl, dashed, "\exists h_2"'] & P^2 \arrow[r] \arrow[d, "f_2"] \arrow[dl, dashed, "\exists h_1"'] & P^1 \arrow[r] \arrow[d, "f_1"] \arrow[dl, dashed, "\exists h_0"'] & P^0 \arrow[r] \arrow[d, "f_0"] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & \ldots \\
P^*(Y)[-1] & \ldots \arrow[r] & Q^4 \arrow[r] & Q^3 \arrow[r] & Q^2 \arrow[r] & Q^1 \arrow[r] & Q^0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}
$$

Both complexes are exact everywhere except for $X \cong P^0/d(P^1)$ and $Y \cong Q^0/d(Q^1)$.

$\Rightarrow \exists h_0: P^0 \to Q^2$ such that f_0 factors through h_0

$\Rightarrow \exists h_1: P^1 \to Q^3$ such that $f_1 - h_0 d$ factors through h_1

$\Rightarrow \exists h_2: P^2 \to Q^4$ such that $f_2 - h_1 d$ factors through h_2, and so on

$\Rightarrow$ f is homotopic to zero

Result: $\mathrm{Hom}_{D(R\text{-Mod})}(X, Y[-1]) = 0$ $\forall X, Y$ modules

(and thus $\mathrm{Hom}_{D(R\text{-Mod})}(X, Y[-n]) = 0$ $\forall X, Y$ modules, $n \in \mathbb{N}$).

In the opposite direction, i.e. $P^*(X) \xrightarrow{f} P^*(Y)[n]$, $n \geq 0$, there are, in general, many non-zero homomorphisms in $K^-(R\text{-Proj})$. For instance, when X is given and $P^*(X)$ has been chosen, and $P^n \neq 0$ for some n, then let $Y = P^n/d(P^{n+1})$, $\pi: P^n \to P^n/d(P^{n+1})$ the projection and f as in the diagram. Then f is a map of complexes and $f \neq 0$ unless $d: P^{n+1} \to P^n$ is surjective.

$$
\begin{tikzcd}
P^*(X) \arrow[d, "f"] & \ldots \arrow[r] & P^{n+1} \arrow[r, "d"] \arrow[d, "0"] \arrow[dr, phantom, "\circlearrowleft"] & P^n \arrow[r, "d"] \arrow[d, "\pi"] \arrow[dr, phantom, "\circlearrowleft"] & P^{n-1} \arrow[r] \arrow[d, "0"] & \ldots \\
P^*(Y)[n] & \ldots \arrow[r] & 0 \arrow[r] & Y = P^n/d(P^n) \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}
$$

$\Rightarrow$ The situation is asymmetric: $\mathrm{Hom}_{D(R\text{-Mod})}(X, Y[n])$ can be non-zero for $n \geq 0$, but not for $n < 0$.

Wasn't there a choice when defining the shift? No! There is choice in notation, for instance when choosing chain complexes or cochain complexes. And later we have to choose the shift we use for verifying the triangulated structure.

And this choice has consequences – it is contents, not just notation:

Recall exercise (2) on the problem sheet "Exercises on triangulated categories":
When $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ is a distinguished triangle and $h: Z \to X[1]$ satisfies $h=0$, then $Y \simeq X \oplus Z$. So, using the shift of complexes in the opposite direction (shifting to the right), exact sequences of modules would lead to split exact triangles, which is not what we want.
$\to$ For our results so far and yet to come it is important that the shift used as auto-equivalence in the triangulated structure moves a complex (written from left to right) to the left. (So, depending on other choices of notation, it sometimes may be called $[-1]$ instead of $[1]$. But the shift operation as such must be this one.)

For other objects X and Y, $\mathrm{Hom}_{D^-(R\text{-Mod})}(X, Y[-1])$ can, of course, be non-zero: Choose for instance $Y := X[+1]$, then $\mathrm{Hom}_{D^-(R\text{-Mod})}(X, Y[-1]) \simeq \mathrm{Hom}_{D^-(R\text{-Mod})}(X,X)$. The vanishing of $\mathrm{Hom}(X, Y[-1])$ in $K^-(R\text{-Mod}) \simeq D^-(R\text{-Mod})$ is a particular property, shared by all modules X, Y and some other objects, but not by all pairs of objects.

Let's continue to do computations: Let $X = P \neq 0$ be projective and Y any module, and $n \in \mathbb{N}$. What is $\mathrm{Hom}_{D^-(R\text{-Mod})}(X, Y[n])$? Let $P^*(Y) = \ldots \to Q^2 \to Q^1 \to Q^0 \to 0 \to \ldots$

$$\begin{array}{ccl} X & & \to 0 \to 0 \to 0 \to 0 \to \ldots \to \underset{\uparrow \text{ place } 0}{X = P} \to 0 \to \ldots \\ f \downarrow & & \\ P^*(Y)[n] & & \ldots \to Q^2 \to Q^1 \to \underset{\uparrow \text{ place } n}{Q^0} \to 0 \to 0 \to \ldots \end{array}$$

$\Rightarrow f = 0$.

Result: $\mathrm{Hom}_{D(R\text{-Mod})}(X = P, Y[n]) = 0 \quad \forall X = P$ projective, Y module, $n \geq 1$.

Next computation: Let X be any module, $P^*(X)$ a projective resolution and $\Omega_1(X)$, $\Omega_2(X)$, ... the syzygies.

$P^*(X): \ldots \to P^{n+1} \to P^n \to P^{n-1} \to \ldots \to P^2 \to P^1 \to P^0 \to 0 \to \ldots \Rightarrow$

$P^*(\Omega_n X)$ = projective resolution of $\Omega_n X$: $\ldots \to P^{n+2} \to P^{n+1} \to P^n \to 0 \to \ldots$

⇝ There is a morphism

$$\begin{array}{ccccccccccccccccc}
P^*(X) & \cdots \to & P^{n+2} & \to & P^{n+1} & \to & P^n & \to & P^{n-1} & \to & \cdots & \to & P^2 \to P^1 \to P^0 \to 0 \to \cdots \\
\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
P^*(\Omega_n X)[n] & \cdots \to & P^{n+2} & \to & P^{n+1} & \to & P^n & \to & 0 & \to & 0 & \to & \cdots
\end{array}$$

which is homotopic to zero exactly when $\Omega_n X = 0$.

Did you start conjecturing a meaning of $\mathrm{Hom}_{D^-(R\text{-Mod})}(X, Y[n])$ for $n \in \mathbb{N}$?

Another piece of evidence comes from Proposition 12.11: Let U, X, Y, Z be R-modules and $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ an exact sequence in R-Mod. Choosing projective resolutions $P^*(X)$ and $P^*(Z)$ and constructing one for Y using the Horseshoe Lemma provides a short exact sequence of cochain complexes

$$0 \to P^*(X) \xrightarrow{f} P^*(Y) \xrightarrow{g} P^*(Z) \to 0$$

and distinguished triangles $P^*(X) \xrightarrow{f} P^*(Y) \xrightarrow{g} P^*(Z) \xrightarrow{h}$ in $K^-(R\text{-Proj})$ and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ in $D^-(R\text{-Mod})$.

By Proposition 12.11, there is a long exact sequence in $D^-(R\text{-Mod})$

$$\cdots \to \underbrace{\mathrm{Hom}(U, Z[-2])}_{=0} \to \underbrace{\mathrm{Hom}(U, X[-1])}_{=0} \to \underbrace{\mathrm{Hom}(U, Y[-1])}_{=0} \to \underbrace{\mathrm{Hom}(U, Z[-1])}_{=0} \longrightarrow$$

$$\longrightarrow \mathrm{Hom}(U, X) \to \mathrm{Hom}(U, Y) \to \mathrm{Hom}(U, Z) \to \mathrm{Hom}(U, X[1]) \to \mathrm{Hom}(U, Y[1]) \to \cdots$$

where, as we know, all the terms with negatively shifted modules are zero. This looks like a cohomological δ-functor, more precisely like the long exact sequence for Ext, in the covariant case. And analogously in the contravariant case:

$$\cdots \to \underset{\underset{0}{\parallel}}{\underset{\mathrm{Hom}(Z, U[-1])}{\underset{\cong}{\mathrm{Hom}(Z[1], U)}}} \xrightarrow{g} \underset{\underset{0}{\parallel}}{\underset{\mathrm{Hom}(Y, U[-1])}{\underset{\parallel}{\mathrm{Hom}(Y[1], U)}}} \xrightarrow{f} \underset{\underset{0}{\parallel}}{\underset{\mathrm{Hom}(X, U[-1])}{\underset{\parallel}{\mathrm{Hom}(X[1], U)}}} \xrightarrow{h}$$

$$\xrightarrow{h} \mathrm{Hom}(Z, U) \xrightarrow{g} \mathrm{Hom}(Y, U) \xrightarrow{f} \mathrm{Hom}(X, U) \xrightarrow{h} \underset{\mathrm{Hom}(Z[-1], U)}{\underset{\parallel}{\mathrm{Hom}(Z, U[1])}} \to \overset{\mathrm{Hom}(Y[-1], U)}{\overset{\parallel}{\underset{0 \text{ if } Y \text{ projective}}{\underset{\parallel}{\mathrm{Hom}(Y, U[1])}}}} \to \cdots$$

If Y happens to be projective, we can compare with the definition of $\mathrm{Ext}^1_{R\text{-Mod}}(Z, U)$ in the abelian case and get:

13.1 Theorem: Let R be a ring and $X, Y \in R\text{-Mod}$. Then
$$\mathrm{Ext}^1_R(X,Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y[1]),$$
where $\mathcal{C}$ can be any of the triangulated categories $D^b(R\text{-Mod})$, $D^-(R\text{-Mod})$, $D^+(R\text{-Mod})$ or $D(R\text{-Mod})$.
When $P^*(X)$ and $P^*(Y)$ are projective resolutions of X and Y, respectively, then
$$\mathrm{Ext}^1_R(X,Y) \simeq \mathrm{Hom}_{K^-(R\text{-Mod})}(P^*(X), P^*(Y)[1]).$$

(Comparing the long exact sequences we also see that h corresponds to the connecting homomorphism. However, since the cone in a distinguished only is unique up to a non-unique isomorphism, we cannot say that h equals the connecting homomorphism.)

If we view a morphism $\varphi: X \to Y[1]$ as a morphism $P^*(X) \longrightarrow P^*(Y)[1]$

$$\begin{tikzcd}
\cdots \arrow[r] & P^2 \arrow[r] \arrow[d] & P^1 \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d] & \operatorname{coker} = X \\
\cdots \arrow[r] & Q^1 \arrow[r] & Q^0 \arrow[r] & 0 &
\end{tikzcd}$$

then we can see the corresponding exact sequence $0 \to Y \to E \to X \to 0$ in the same way as we did in chapter 3:

$$\begin{tikzcd}
P^1 \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[dd] & X \arrow[dd, equal] \\
Q^0 \arrow[d] & & \\
Y \arrow[r] & E \arrow[r] & X
\end{tikzcd}$$

E = pushout

Theorem 13.1 suggests to define in any triangulated category $\mathcal{T}$
$$\mathrm{Ext}^i_{\mathcal{T}}(X,Y) := \mathrm{Hom}_{\mathcal{T}}(X, Y[i])$$
which now just is a notation, extending the definition we know for abelian categories.

$D^b(R\text{-Mod})$ always contains infinitely many pairwise non-isomorphic indecomposable objects: Choose M an indecomposable R-module, then $\{M[n] : n \in \mathbb{Z}\}$ is a class of examples. The $M[n]$ are indecomposable since $\mathrm{End}_{D^b(R\text{-Mod})}(M[n]) \simeq \mathrm{End}_{D^b(R\text{-Mod})}(M) = \mathrm{End}_R(M)$.

Are there other indecomposable objects not of this form?

Let M be an indecomposable module of projective dimension at least two and $\dots \to P^2 \to P^1 \to P^0 \to 0 \to \dots$ a minimal projective resolution of M.
Then $\dots \to 0 \to P^1 \to P^0 \to 0 \to \dots$ is indecomposable why?
and has non-zero cohomology in degrees 1 and 0 why?
So it cannot be isomorphic to a stalk complex $\dots \to 0 \to X \to 0 \to \dots$ in $D^b(R\text{-Mod})$
$\Rightarrow$ It can only happen, if at all, for hereditary rings that all indecomposable complexes are isomorphic to stalk complexes.

13.2 Theorem: Let $\mathcal{A}$ be a hereditary abelian category, i.e. $\mathrm{Ext}^2_{\mathcal{A}}(-,-)$ is defined and vanishes always, such that $D(\mathcal{A})$ (or $D^-(\mathcal{A})$ or $D^b(\mathcal{A})$) is defined.
Then every object X^* in the derived category is isomorphic to a cochain complex with differential 0:

$$X^* \simeq \dots \xrightarrow{0} H^{n-1}X \xrightarrow{0} H^n X \xrightarrow{0} H^{n+1}X \to \dots = H^*$$

In particular, every indecomposable object in the derived category is a stalk complex (with cohomology concentrated in one degree, or just 0).

Proof: $X^* = \dots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \dots$
There is little chance to find maps of complexes $X^* \to H^*$ or $H^* \to X^*$ that turn out to be quasi-isomorphisms. Instead we will find q.i.s $X^* \leftarrow Y^* \to H^*$ for some Y^*.
For each n there are short exact sequences in $\mathcal{A}$:

$$0 \to \mathrm{Im}(d^{n-1}) \to \mathrm{Ker}(d^n) \to H^n X \to 0 \quad (*)$$

$$\text{and} \quad 0 \to \mathrm{Ker}(d^{n-1}) \to X^{n-1} \to \mathrm{Im}(d^{n-1}) \to 0$$

In order to use the assumption $Ext^2_{\mathcal{A}}(-,-)=0$ we write a long exact cohomology sequence:

$$0 \to Hom_{\mathcal{A}}(H^nX, Ker(d^{n-1})) \to Hom_{\mathcal{A}}(H^nX, X^{n-1}) \to Hom_{\mathcal{A}}(H^nX, Im(d^{n-1})) \to$$
$$\to Ext^1_{\mathcal{A}}(H^nX, Ker(d^{n-1})) \to Ext^1_{\mathcal{A}}(H^nX, X^{n-1}) \to Ext^1_{\mathcal{A}}(H^nX, Im(d^{n-1})) \to$$
$$\to Ext^2 = 0$$

$\Rightarrow$ surjective $\qquad (*) \in Ext^1_{\mathcal{A}}(H^nX, Im(d^{n-1}))$

$\Rightarrow$ The ses $(*)$ has a preimage

$$\begin{tikzcd}
(+) & 0 \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & E^n \arrow[r] \arrow[d] & H^nX \arrow[r] \arrow[d, equal] & 0 \quad (\text{for some } E^n \text{ in } \mathcal{A}) \\
(*) & 0 \arrow[r] & Im(d^{n-1}) \arrow[r] & Ker(d^n) \arrow[r] & H^nX \arrow[r] & 0
\end{tikzcd}$$

pushout

We can put $(+)$ into a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^nX \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & X^{n-1} \arrow[r] \arrow[u] \arrow[d] & E^n \arrow[r] \arrow[u] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & \cdots \\
\cdots \arrow[r] & X^{n-2} \arrow[r] & X^{n-1} \arrow[r, "d^{n-1}"] & X^n \arrow[r, "d^n"] & X^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

check commutativity

where the vertical maps induce isomorphisms of cohomology in degree n.

This suggests to define Y^k as

$$\cdots \to X^{n-2} \oplus E^{n-2} \to X^{n-1} \oplus E^{n-1} \to X^n \oplus E^n \to \cdots$$

Such that in the next diagram all vertical maps induce isomorphisms in cohomology, in every degree:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-2}X \arrow[r, "0"] & H^{n-1}X \arrow[r, "0"] & H^nX \arrow[r, "0"] & H^{n+1}X \arrow[r] & \cdots \\
\cdots \arrow[r] & X^{n-2} \oplus E^{n-2} \arrow[r] \arrow[u] \arrow[d] & X^{n-1} \oplus E^{n-1} \arrow[r] \arrow[u] \arrow[d] & X^n \oplus E^n \arrow[r] \arrow[u] \arrow[d] & X^{n+1} \oplus E^{n+1} \arrow[r] \arrow[u] \arrow[d] & \cdots \\
\cdots \arrow[r] & X^{n-2} \arrow[r] & X^{n-1} \arrow[r] & X^n \arrow[r] & X^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

fill in the maps

$\square$