Let us return to the list of tasks on page 12.13. The first two items have been done: We have defined triangulated categories and we have shown that homotopy categories $K(\mathcal{A})$ are triangulated. The next task is to check that quasi-isomorphisms in $K(\mathcal{A})$ provide multiplicative systems.

More generally:

12.16 Proposition: Let $\mathcal{T}$ be a triangulated category, $\mathcal{A}$ an abelian category, $H: \mathcal{T} \longrightarrow \mathcal{A}$ a cohomological functor and S the class of all morphisms s in $\mathcal{T}$ such that $H^i(s)$ is an isomorphism for all $i \in \mathbb{Z}$. Then S is a multiplicative system.

(For instance, choosing $H = H^0$ yields the quasi-isomorphisms.)

Proof: Multiplicative systems have been defined in Definition 12.6. Identity morphisms are in S and S is closed under products, since H is a functor. $\Rightarrow$ (1) ✓

(2) To check the Ore condition, let
$$\begin{array}{ccc} & & Z \\ & & \downarrow t \\ X & \xrightarrow{s} & Y \end{array}$$
be given and $t \in S$. $Z \xrightarrow{t} Y$ can be embedded into a distinguished triangle $Z \xrightarrow{t} Y \xrightarrow{u} C \rightsquigarrow Z[1]$.

The composition $u \circ g$ also can be embedded into a distinguished triangle
$$X \xrightarrow{u \circ g} C \longrightarrow W \rightsquigarrow X[1]$$
and after rotating $W[-1] \longrightarrow X \xrightarrow{u \circ g} C \rightsquigarrow W \rightsquigarrow$ There is a commutative diagram
$$\begin{tikzcd} W[-1] \arrow[r, "s"] \arrow[d, dashed, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[r, "u \circ g"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r, squiggly, "v"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, dashed, "h[1]"] \\ Z \arrow[r, "t"] & Y \arrow[r, "u"] & C \arrow[r, squiggly] & Z[1] \end{tikzcd}$$
By (TR3), there exists h such that the squares commute.

It remains to check that $s \in S$.

By assumption, $t \in S$, hence all $H^i(t)$ are isomorphisms. Since H is cohomological, the isomorphisms $H^i(t)$ force $H^i(C)$ to vanish for all $i \in \mathbb{Z}$. Now applying H to the top triangle, all $H^i(C) = 0$ forces all $H^i(s)$ to be isomorphisms. $\Rightarrow s \in S$.

The symmetric statement in (2) follows in the same way.

(3) Finally we have to check the cancellation rule:

Let $f, g : X \to Y$ in $\mathcal{T}$ be given. $\mathcal{T}$ is additive $\Rightarrow h := f-g$ is defined. Suppose (a): $s \circ f = s \circ g$ for some $s \in S$, for $s: Y \to Y'$. We have to find $t \in S$ such that $f \circ t = g \circ t$.
Embed s into a distinguished triangle (and rotate it):

$$Z \xrightarrow{u} Y \xrightarrow{s} Y' \rightsquigarrow Z[1]$$

Because of $s \in S$, $H^i(Z) = 0 \; \forall i \in \mathbb{Z}$.
$\mathrm{Hom}_{\mathcal{T}}(X,-)$ is a cohomological functor by 12.11, hence exact $\Rightarrow$

$$\mathrm{Hom}_{\mathcal{T}}(X,Z) \xrightarrow{u} \mathrm{Hom}_{\mathcal{T}}(X,Y) \xrightarrow{s} \mathrm{Hom}_{\mathcal{T}}(X,Y') \text{ is exact}$$

$s \circ h = s \circ (f-g) = 0 \Rightarrow h$ is in the image of u, i.e. $\exists j : X \to Z$ such that $h = u \circ j$
$\exists$ distinguished triangle containing j: $X' \xrightarrow{t} X \xrightarrow{j} Z \rightsquigarrow X'[1]$
Here, $0 = j \circ t$ implies $h \circ t = u \circ j \circ t = 0$, ie $(f-g) \circ t = 0$ or $f \circ t = g \circ t$.
It remains to check that $t \in S$: $H^i(Z) = 0 \; \forall i \in \mathbb{Z}$ implies, using the long exact sequence for H, that all $H^i(t)$ are isomorphisms. $\Rightarrow t \in S$.
The proof of (b) $\Rightarrow$ (a) is similar. □

This proof illustrates how useful the language of triangulated categories can be.

At this point, Definition 12.7 of the derived category is now justified, apart from potential set-theoretic problems: $D(\mathcal{A}) := K(\mathcal{A})[Qis^{-1}]$ makes sense.

Next we turn our attention to localisation preserving triangulated structures, which in particular puts a triangulated structure on $D(\mathcal{A})$.
We stick to S the class of quasi-isomorphisms associated with a cohomological functor H as in 12.16. For instance, $S = Qis$ as in 12.7 is allowed. Such an S is compatible with the triangulated structure in the following sense:
- for $s \in S$, $s[n] \in S \; \forall n \in \mathbb{Z}$
- if (f, g, h) is a morphism between distinguished triangles and $f, g \in S$, then $\exists$ morphism $h' \in S$ such that (f, g, h') is a morphism between these distinguished triangles.

(Localising at a multiplicative system that is not compatible with the triangulated structure would cause problems.)

To define a triangulated structure on $\mathcal{T}[S^{-1}]$, we have to define the shift functor and the distinguished triangles, and then verify the axioms. Since $\mathcal{T}[S^{-1}]$ has the same objects as $\mathcal{T}$, the shift on objects is just that of $\mathcal{T}$. A morphism in $\mathcal{T}[S^{-1}]$ is given by a roof, for instance $\begin{tikzcd}[column sep=small] & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y' \end{tikzcd}$, which we view as a fraction $f \circ s^{-1}$.

In $\mathcal{T}$, the shifts $T(f)$ and $T(s)$ are defined. As S is compatible with the triangulation, $T(s) = s[1] \in S \Rightarrow$ In $\mathcal{T}[S^{-1}]$, $T(s)^{-1}$ exists and we can define $T(\frac{f}{s}) := T(s)^{-1} \circ T(f)$, and similarly for right fractions.
$\Rightarrow T: \mathcal{T}[S^{-1}] \longrightarrow \mathcal{T}[S^{-1}]$ is defined.
Next we define distinguished triangles in $\mathcal{T}[S^{-1}]$: Given objects X, Y and Z and morphisms

$$(*) \quad \begin{tikzcd}[column sep=small] & X' \arrow[dl, "s_1"'] \arrow[dr, "f"] & & Y' \arrow[dl, "s_2"'] \arrow[dr, "g"] & & Z' \arrow[dl, "s_3"'] \arrow[dr, "h"] & \\ X & & Y & & Z & & X[1] \end{tikzcd}$$

we use the Ore condition to rewrite fractions so that we get the following diagram

$$\begin{tikzcd}[column sep=small] & & & X''' \arrow[dl, "s_1''"'] \arrow[dr, "f''"] & & & \\ & & X'' \arrow[dl, "s_1'"'] \arrow[dr, "f'"] & & Y'' \arrow[dl, "s_2'"'] \arrow[dr, "g'"] & & \\ & X' \arrow[dl, "s_1"'] \arrow[dr, "f"] & & Y' \arrow[dl, "s_2"'] \arrow[dr, "g"] & & Z' \arrow[dl, "s_3"'] \arrow[dr, "h"] & \\ X & & Y & & Z & & X[1] \end{tikzcd}$$

where all s_i', s_i'' are in S and hence in $\mathcal{T}[S^{-1}]$, $X \simeq X' \simeq X'' \simeq X'''$, $Y \simeq Y' \simeq Y''$ and $Z \simeq Z'$. We call $(*)$ a distinguished triangle in $\mathcal{T}[S^{-1}]$ if $X''' \xrightarrow{f''} Y'' \xrightarrow{g'} Z' \xrightarrow{h} X[1]$ is a distinguished triangle in $\mathcal{T}$. As usual, everything isomorphic to $(*)$ gets a distinction as well.
To check that $\mathcal{T}[S^{-1}]$ satisfies all the required axioms, one has to rewrite all the distinguished triangles in $\mathcal{T}[S^{-1}]$ as above and then apply the axioms valid in $\mathcal{T}$. We omit that.
Now we know that $D(\mathcal{A})$ is a triangulated category – if it exists.

This brings us to the last problem remaining. When is $\mathcal{J}[S^{-1}]$ a category, or more precisely, when do $\mathrm{Hom}_{\mathcal{J}[S^{-1}]}(X,Y)$ always form sets, for X,Y objects in $\mathcal{J}$. In the literature there is a tendency to gloss over this problem or to "solve" it by the handwaving suggestion to increase the universe, if necessary (which means: redefine sets so that all $\mathrm{Hom}_{\mathcal{J}[S^{-1}]}(X,Y)$ are sets according to the new definition).

What is the problem exactly? Given a category $\mathcal{C}$, the objects may not form a set, and for a given object X, the class $\{X' \in \mathrm{Ob}\,\mathcal{C} : X' \cong X\}$ need not be a set. This happens already for $\mathcal{C} = k\text{-mod}$, the finite dimensional vectorspaces. Therefore, for fixed X and Y, the roofs $X \xleftarrow{s} X' \xrightarrow{f} Y$ may not form a set. In the examples we are interested, such as $S = \mathrm{Qis}$, S is closed under composition with isomorphisms, and then the roofs $X \xleftarrow{s\alpha} \tilde{X} \xrightarrow{f} Y$ have no chance to form a set.

For $\mathcal{C} = k\text{-mod}$ we can solve the problem, as we did in linear algebra, by considering the full subcategory $\mathcal{C}'$ with objects $\{k^n : n \in \mathbb{N}_0\}$, which by inclusion is equivalent to $\mathcal{C}$. $\mathcal{C}'$ is so small that everything of interest forms a set. This approach is, however, restricted to special situations. In general, we should try to use that we don't need the roofs $X \leftarrow X' \rightarrow Y$ to form a set: $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X,Y)$ consists of equivalence classes, and only these need to form a set. This can be used when the following condition is satisfied:

A multiplicative system S is locally small (on the left) if for each object X there exists a set S_X of morphisms $Y \xrightarrow{s} X$, all having target X, such that for every $s' : Y' \to X$, $s' \in S$ $\exists\, Y \xrightarrow{s} X$ in S and a factorisation

$$\begin{tikzcd} Y' \arrow[rr, "s'"] & & X \\ & Y \arrow[ul, "h"] \arrow[ur, "s \in S_X"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

so by expanding fractions we get denominators in S_X.

A non-trivial proof then shows how to define $\mathcal{C}[S^{-1}]$ by just using S_X, and then $\mathcal{C}[S^{-1}]$ (or $[S^{-1}]\mathcal{C}$) is a category, which has the localisation property. This is a theorem of Gabriel–Zisman.

When R is any ring, one can prove that $S = \mathrm{Qis}$ is a locally small multiplicative system in $K(R\text{-Mod})$. Hence $D(R\text{-Mod})$ does exist.

How does one prove that? We don't go into all details, but collect what seems to be the main ideas to be used:

• An R-module is a set X with additional structure. Any R-module of cardinality $\mathrm{card}(X)$ defines a module structure on X. Thus, counting isomorphism classes of R-modules of a given cardinality $\mathrm{card}\,X$ is the same as counting module structures on X itself. Since X and R are sets, the functions $R \times X \to X$ form a set.

$\Rightarrow$ The isomorphism classes of R-modules of cardinality $\mathrm{card}(X)$ form a set.

• Let Y and X be R-modules and $f: Y \to X$ a surjective R-module homomorphism. Then Y has a submodule Y_0 of cardinality $\mathrm{card}(Y_0) \leq \mathrm{card}(R) \cdot \mathrm{card}(X)$ such that the restriction $f|_{Y_0}: Y_0 \to X$ is surjective. In fact, choose a set of generators of X, for instance X itself. Then $\exists \bigoplus_{g \text{ generator}} R \twoheadrightarrow X$, where 1_R in summand g gets sent to $g \in X$.

This factors
$$\begin{tikzcd} & \bigoplus R \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ Y \arrow[r, "f"'] & X \arrow[r] & 0 \end{tikzcd}$$
Then $Y_0 :=$ Image of h works.

• Let $\mathcal{C} \in Ch(R\text{-Mod})$ a cochain complex, $\mathcal{C}: \ldots \to C^{-1} \to C^0 \to C^1 \to \ldots$ and choose an infinite cardinal c_0 that is bigger than $\mathrm{card}(R)$ and the cardinality of any C^i.

The cochain complexes $\mathcal{C}'$ that are in that sense also smaller than c_0 form isomorphism classes, and these isomorphism classes form a set.

To show that Q is locally small, one reduces the situation to quasi-isomorphisms $\mathcal{C}' \to \mathcal{C}$. One has to show that for cochain complexes $\mathcal{D}$ that are larger and qis $\mathcal{D} \to \mathcal{C}$ one finds a factorisation
$$\begin{tikzcd} \mathcal{D} \arrow[rr, "f"] & & \mathcal{C} \\ & \mathcal{C}' \arrow[ul] \arrow[ur, "s"'] & \end{tikzcd}$$
where s is a qis, too.

$\mathcal{C}'$ is constructed as a subcomplex of $\mathcal{D}$.

$H^*(f): H^*(\mathcal{D}) \to H^*(\mathcal{C})$ is an isomorphism. $H^i = \mathrm{Ker}(d)/\mathrm{Im}(d)$ in each complex.

$\ldots 0 \to 0 \to 0 \to \underset{\text{degree } i}{\mathrm{Ker}(d)} \to 0 \to \ldots$ is a subcomplex. If $\mathrm{Ker}(d)$ is too large (in $\mathcal{D}$), it contains a submodule K' such that K' still maps surjectively onto $H^i(\mathcal{C})$, and $\mathrm{card}(K') < c_0$. $\Rightarrow \exists$ subcomplex of $\mathcal{D}$ with all terms of cardinality less than c_0 such that f induces a surjection from its homology onto $H^*(\mathcal{C})$. This surjection may not be an isomorphism, but the kernel of the surjection has, of course, also small enough cardinality. Thus one can inductively repair the problem and eventually find $\mathcal{C}'$!

Another way to prove something is a category, is to show that it is equivalent to a category. This works for some variants of $D(\mathcal{A})$ under assumptions on $\mathcal{A}$:

<u>12.17 Theorem</u>: Let $\mathcal{A}$ be an abelian category with enough projectives. Then the restriction of the natural functor $K^-(\mathcal{A}) \to D^-(\mathcal{A})$ induces an equivalence of triangulated categories $K^-(\mathcal{A}\text{-Proj}) \xrightarrow{\sim} D^-(\mathcal{A})$, which furter restricts to an equivalence $K^{-,b}(\mathcal{A}\text{-Proj}) \xrightarrow{\sim} D^b(\mathcal{A})$.
When $\mathcal{A}$ has enough injectives, analogous results hold true for $D^+(\mathcal{A})$ and $D^{+,b}(\mathcal{A})$.

In the proof we will identify Hom in $K^-(\mathcal{A}\text{-Proj})$ with Hom in $D^-(\mathcal{A})$, which proves that the latter are sets as well.
Alternatively, 12.17 also allows to define $D^-(\mathcal{A}) := K^-(\mathcal{A}\text{-Proj})$ and check that this category has the defining property of the localisation at quasi-isomorphisms.

Proof of 12.17: Inclusion $K^-(\mathcal{A}\text{-Proj}) \hookrightarrow K^-(\mathcal{A})$ and localisation $K^-(\mathcal{A}) \to D^-(\mathcal{A})$ are triangulated functors, hence the composition is so as well. We have to show that this composition is full, faithful and dense.
Let us first check a special case, to get an idea: What happens for objects X and Y in $\mathcal{A}$? They are objects in $D^-(\mathcal{A})$. In $K^-(\mathcal{A}\text{-Proj})$, there is an object P^*, the projective resolution of X, which in $K^-(\mathcal{A})$ and in $D^-(\mathcal{A})$ is isomorphic to X. P^* has homology $X = H^0(P^*)$.
Similarly, Y has a projective resolution Q^* with $Y = H^0(Q^*)$. P^* and Q^* are unique up to homotopy, hence unique as objects of $K^-(\mathcal{A}\text{-Proj})$. A morphism $f: P^* \to Q^*$ of complexes induces a morphism $\hat{f}: H^0(P^*) = X \to H^0(Q^*) = Y$. Conversely, a map $X \to Y$ in $\mathcal{A}$ lifts to a map $P^* \to Q^*$, unique up to homotopy. Moreover, $f: P^* \to Q^*$ is qis iff $\hat{f}$ is an isomorphism. So, this looks good.
Now we consider the general case and show that $K^-(\mathcal{A}\text{-Proj}) \to K^-(\mathcal{A}) \to D^-(\mathcal{A})$ is full, faithful and dense.
Faithfull: Inclusion $K^-(\mathcal{A}\text{-Proj}) \to K^-(\mathcal{A})$ is of course faithful. The question is what happens when quasi-isomorphisms become invertible.

Let $f: X^\cdot \to Y^\cdot$ be a quasi-isomorphism between objects in $K^-(\mathcal{A}\text{-Proj})$.
By 12.15, $K^-(\mathcal{A})$ is a triangulated $\Rightarrow \exists$ distinguished triangle $X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \rightsquigarrow$
where we can choose $Z^\cdot$ to be the mapping cone. Since f is a q.is, $H^n(Z^\cdot) = 0 \ \forall n$.
So, $Z^\cdot$ is a bounded above complex whose terms are projective and that is exact.
By induction $Z^\cdot$ can be shown to be trivial in the homotopy category (the induction starts needs that $Z^\cdot$ is bounded above) $\Rightarrow Z^\cdot \simeq 0$ and by 12.8 f is an isomorphism in $K^-(\mathcal{A})$, and also in $K^-(\mathcal{A}\text{-Proj})$. (This also is an instance of the Comparison Theorem for projective resolution.)
This tells us that nothing happens when making q.is invertible, since they are already invertible.

This implies faithful. It also implies full, provided we can show dense without using full.
Dense: Let $X^\cdot$ be an object in $D^-(\mathcal{A})$. We have to show it is isomorphic to the image of an object in $K^-(\mathcal{A}\text{-Proj})$. So we have to find a kind of projective resolution of $X^\cdot$, that is, a bounded above complex with projective terms and the same cohomology as $X^\cdot$.
There is a general procedure to construct such a complex, called Cartan-Eilenberg resolution.
The naive idea is to resolve the terms of $X^\cdot$ separately: Here $P, P', \ldots, Q, Q', \ldots, R, R'$ form projective resolutions, and the horizontal maps lift those in $X^\cdot$.

$$\begin{tikzcd}
 & R' \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & P' \arrow[d] & \\
 & R \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & P \arrow[d] & \\
\cdots \arrow[r] & X^{-2} \arrow[r] & X^{-1} \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Then one adds up "diagonally" to form the "double complex" into $\to R \oplus Q' \oplus P'' \to Q \oplus P' \to P \to 0$, using the given differentials, modified by new signs.
Unfortunately, this is too naive, since it doesn't take into account the homology of $X^\cdot$. Instead one should resolve boundaries B^n and cohomology H^n separately and lift it (by the Horseshoe Lemma) to resolutions of Z^n and then, again using the Horseshoe Lemma, also of X^n. And then form the diagonal sums. This is a bit tedious, but it works. (When $X^\cdot$ is an unbounded complex, the diagonal sums are infinite sums, whose existence one has to assume. We don't need these general resolutions.)
We are in a less general situation ($X^\cdot$ is bounded above) and can do an ad hoc construction, proceeding inductively:

Let $X^\bullet$ be a (non-zero) complex in $D^-(\mathcal{A})$, bounded above. In particular, $H^i(X^\bullet)=0$ for large i. By shifting $X^\bullet$ we may assume $H^0(X^\bullet)\neq 0$ and $H^i(X^\bullet)=0$ $\forall i>0$.
Still it may happen that $X^i \neq 0$ for $i>0$: $X^\bullet = \ldots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \to \ldots$
In that case, let $\tau_{\leq 0} X^\bullet$ be a truncation of $X^\bullet$:

$$\begin{tikzcd}
\tau_{\leq 0} X^\bullet := \ldots \arrow[r] & X^{-2} \arrow[r, "d^{-2}"] \arrow[d, equal] & X^{-1} \arrow[r, "d^{-1}"] \arrow[d, equal] & \mathrm{Ker}(d^0) \arrow[r, "0"] \arrow[d, hook] & 0 \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d, hook] & \ldots \\
\ldots \arrow[r] & X^{-2} \arrow[r, "d^{-2}"] & X^{-1} \arrow[r, "d^{-1}"] & X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & \ldots
\end{tikzcd}$$

which is a subcomplex of $X^\bullet$:

The inclusion $\tau_{\leq 0}(X^\bullet) \to X^\bullet$ is a morphism of complexes and it is even a quasi-isomorphism. why?
Therefore we may and will assume from now on that $X^i=0$ $\forall i>0$, replacing if necessary $X^\bullet$ by $\tau_{\leq 0}(X^\bullet)$.
When $H(X^\bullet)$ is concentrated in degree 0, then $X^\bullet$ is quasi-isomorphic to $H^0(X^\bullet)$, which is an object in $\mathcal{A}$ and hence has a projective resolution.
In general, the "projective resolution" of $X^\bullet$ is not obvious. We construct such a complex by induction, given $X^\bullet$:

$$\begin{tikzcd}
\ldots \arrow[r] & X^{-2} \arrow[r] & X^{-1} \arrow[r, "d^{-1}"] & X^0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \ldots \\
 & & Q^{-1} \arrow[u] \arrow[r, "\delta^{-1}"] & P^0 \arrow[u] & & &
\end{tikzcd}$$

Choose P^0 projective, mapping onto X^0 and let Q^{-1} be the pullback of

$$\begin{tikzcd}
X^{-1} \arrow[r] & X^0 \\
 & P^0 \arrow[u]
\end{tikzcd}$$

By properties of the pullback (Lemma 3.4), there is a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(d^{-1}) \arrow[r] & X^{-1} \arrow[r] & \mathrm{Im}(d^{-1}) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\delta^{-1}) \arrow[r] \arrow[u, "\cong"] & Q^{-1} \arrow[r] \arrow[u] & \mathrm{Im}(\delta^{-1}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with an isomorphism $\mathrm{Ker}(d^{-1}) = \mathrm{Ker}(\delta^{-1})$, inducing an isomorphism

$$\underset{H^0(X^\bullet)}{\underset{\shortparallel}{\mathrm{Coker}(d^{-1})}} = \underset{H^0(P^\bullet)}{\underset{\shortparallel}{\mathrm{Coker}(\delta^{-1})}}$$

(where $P^\bullet$ is yet to be defined, it will end with $P^0 \to 0$)

Since $d^{-2}: X^{-2} \to X^{-1}$ has image inside $\overset{\mathrm{Ker}(\delta^{-1})}{\overset{\shortparallel}{\mathrm{Ker}(d^{-1})}}$, we get the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & X^{-3} \arrow[r] \arrow[d, equal] & X^{-2} \arrow[r, "d^{-2}"] \arrow[d, equal] & \mathrm{Ker}(d^{-1}) \arrow[r] \arrow[d, equal] & \mathrm{Ker}(d^{-1})/\mathrm{Im}(d^{-2}) = H^{-1}(X^\bullet) \arrow[d, equal] \\
\ldots \arrow[r] & X^{-3} \arrow[r] & X^{-2} \arrow[r, "d^{-2}"] & \mathrm{Ker}(\delta^{-1}) \arrow[r] & \mathrm{Ker}(\delta^{-1})/\mathrm{Im}(d^{-2})
\end{tikzcd}$$

Thus the complexes and

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-3} \arrow[r] & X^{-2} \arrow[r] & X^{-1} \arrow[r, "d^{-1}"] & X^{0} \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & X^{-3} \arrow[r] \arrow[u, equal] & X^{-2} \arrow[r] \arrow[u, equal] & Q^{-1} \arrow[r, "\delta^{-1}"] \arrow[u] & P^{0} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u, equal] & \cdots
\end{tikzcd}$$

have the same cohomology and are isomorphic. Now we choose a projective object P^{-1} mapping onto Q^{-1}, set $Q^{-2} :=$ pullback, and continue.

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-3} \arrow[r] & X^{-2} \arrow[r] & Q^{-1} \arrow[r, "\delta^{-1}"] & P^{0} \arrow[r] & 0 \arrow[r] & \cdots \\
 & & Q^{-2} \arrow[r] \arrow[u] & P^{-1} \arrow[r, "\hat{\delta}^{-1}"] \arrow[u, two heads] & P^{0} \arrow[r] \arrow[u, equal] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

The image of $\hat{\delta}^{-1}$ coincides with $\operatorname{Im}(\delta^{-1})$, hence cohomology in degree does not change, and induction continues. $\square$

There is no direct analogue of 12.17 for unbounded derived categories. There is however additional structure available – model categories – which allows to realise unbounded derived module categories as homotopy categories. This works more generally for unbounded derived categories of abelian categories satisfying additional conditions (Grothendieck categories), but not in full generality and not for all variants of derived categories.

Another approach is to use a very powerful and very abstract "higher structure" called derivators.