Now we are going to define triangulated categories. What should we expect? The category of complexes – $Ch(\mathcal{A})$ – is an abelian category. The homotopy category – $K(\mathcal{A})$ – is not; there are no kernels, cokernels, exact sequences. So, what happens to a short exact sequence of complexes when we view it in $K(\mathcal{A})$ – i.e. same objects, residue classes of morphisms? The structure of a triangulated category is supposed to capture what happens in this situation. A triangulated structure is in some respects weaker than an abelian one, but it is much stronger than just an additive structure.

12.8 Definition: Let $\mathcal{T}$ be an additive category with an auto-equivalence $T = [1] = \Sigma : \mathcal{T} \to \mathcal{T}$, $X \mapsto T(X) = X[1] = \Sigma X$ (called shift or suspension) and a class of sixtuples $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$, where X, Y, Z are objects and f, g, h are morphisms $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ and $Z \xrightarrow{h} X[1]$ (called distinguished triangles). Then $\mathcal{T}$ is a triangulated category if it satisfies the following axioms:

(TR1) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ that is isomorphic to a distinguished triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "g"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[r, "h"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & X[1] \arrow[d, "\wr"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

is itself distinguished.

The triangle $X \xrightarrow{id} X \xrightarrow{0} 0 \xrightarrow{0} X[1]$ is distinguished, for each X.

Each morphism $X \xrightarrow{f} Y$ occurs in some distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \qquad (\text{"there is a triangle above } f\text{"}).$$

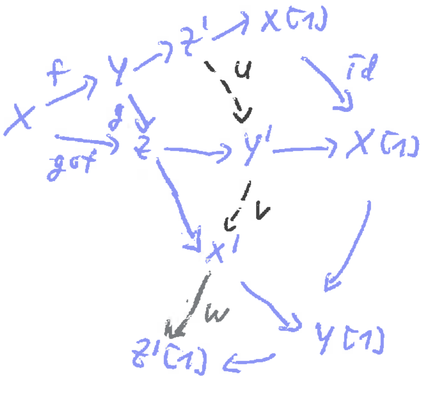
(TR2) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle, then $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ and $Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$ are distinguished triangles.

(TR3) If there is a commutative diagram with two distinguished triangles as rows

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, dashed, "\exists \mu"] & X[1] \arrow[d, "\alpha[1]"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & X'[1] \end{tikzcd}$$

then there exists $\mu : Z \to Z'$ making the diagram commutative, i.e. a morphism of triangles.

(TR4) (the octahedral axiom)



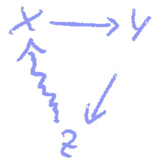
Given $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, complete to distinguished triangles. Then there exist $Z' \xrightarrow{u} Y'$ and $Y' \xrightarrow{v} X'$ such that $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w}$ is a distinguished triangle, and the diagram commutes.

(Here, w is defined by commutativity.)

Triangulated categories have been defined by Verdier in 1963, and also by Dold and Puppe, 1961.

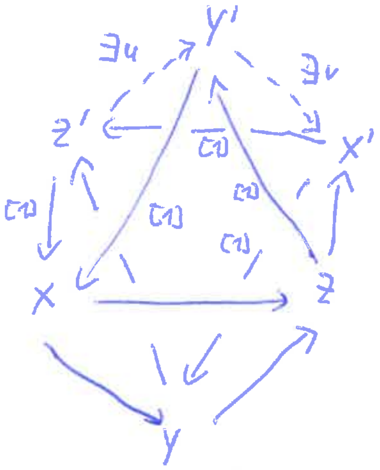
In the literature, different numberings and formulations are used.

One often marks the "degree 1" map $h: Z \to X[1]$ in a distinguished triangle by a different font or style, eg $Z \rightsquigarrow$ or $Z \mapsto$. Then one can write it as a triangle:



Or one uses $\xrightarrow{[1]}$

The octahedral axiom can be written as an octahedron



This picture explains the name, but it is hard to remember.

The diagram in the above statement of (TR4) can be remembered as a kind of third isomorphism theorem: Suppose there are ses in a module category:

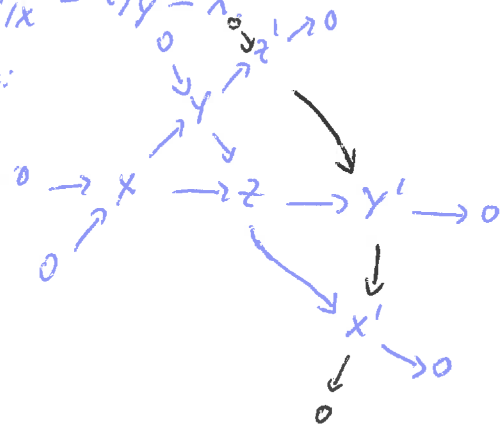
$0 \to X \to Y \to Z' \to 0$, ie $Z' = Y/X$

$0 \to X \to Z \to Y' \to 0$, ie $Y' \simeq Z/X$

$0 \to Y \to Z \to X' \to 0$, ie $X' \simeq Z/Y$

Then $Y'/Z' \simeq {}^{Z/X}\!/_{Y/X} \simeq Z/Y \simeq X'$

Written as a diagram:



We are going to practise working with the axioms by proving some properties of triangulated (abbreviation: Δed) categories.

12.9. Proposition: Let $\mathcal{T}$ be a triangulated category.

(a) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{h}{\rightsquigarrow} X[1]$ be a distinguished triangle. Then $g\circ f = 0$ and $h\circ g = 0$.

(b) Let $X \xrightarrow{f} Y$ be a morphism. Then

(I) f is an isomorphism $\Leftrightarrow$ $X \xrightarrow{f} Y \to 0 \rightsquigarrow X[1]$ is a distinguished triangle.

(II) f is split mono $\Leftrightarrow$ $\exists$ distinguished triangle $X \xrightarrow{f} Y \to Z \overset{0}{\rightsquigarrow} X[1]$

(III) f is split epi $\Leftrightarrow$ $\exists$ distinguished triangle $X \xrightarrow{f} Y \xrightarrow{0} Z \rightsquigarrow X[1]$

Proof: (a) (TR1) $\Rightarrow$ $\exists$ distinguished triangle $X \xrightarrow{1_X} X \to 0 \rightsquigarrow X[1]$

The diagram
$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r, squiggly] \arrow[d, dashed, "\exists\alpha"] & X[1] \arrow[d, "1_X[1]"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, squiggly, "h"] & X[1] \end{tikzcd}$$
commutes $\overset{(TR3)}{\Rightarrow}$ $\exists \alpha: 0 \to Z$ so that all squares commute $\Rightarrow$ $g\circ f = \alpha \circ 0 = 0$

(TR2) allows to rotate the triangle $\Rightarrow$ $h\circ g = 0$ as well.

(b) needs more work.

Claim: (II) and (III) $\Rightarrow$ (I): Let $f: X \to Y$ be an isomorphism. By (TR1) we have to show that $X \to Y \to 0 \rightsquigarrow X[1]$ is isomorphic to a known distinguished triangle:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d, "f^{-1}"] & 0 \arrow[r, squiggly] \arrow[d, equal] & X[1] \arrow[d, equal] \\ X \arrow[r, "\mathrm{id}"] & X \arrow[r] & 0 \arrow[r, squiggly] & X[1] \end{tikzcd}$$
is distinguished by (TR1)

Conversely, suppose $X \xrightarrow{f} Y \to 0 \rightsquigarrow X[1]$ is a distinguished triangle $\Rightarrow$ by (II), f is split mono and by (III), f is split epi $\Rightarrow$ f has a left inverse and a right inverse

$\Rightarrow$ f is an isomorphism

Now we prove (II). (III) has a similar proof, which is omitted.

"$\Leftarrow$" Let $X \xrightarrow{f} Y \to Z \overset{0}{\rightsquigarrow} X[1]$ be a distinguished triangle. To show that f is split mono, we have to find $g: Y \to X$ such that $g\circ f = \mathrm{id}_X$. $\rightsquigarrow$ Compare with another distinguished triangle
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, dashed, "\exists g"] & Z \arrow[r, squiggly, "0"] \arrow[d, "0"] & X[1] \arrow[d, equal] \\ X \arrow[r, "\mathrm{id}"] & X \arrow[r] & 0 \arrow[r, squiggly] & X[1] \end{tikzcd}$$
Rotating, by (TR2), we are allowed to apply (TR3).

$\Rightarrow$ $\exists g: Y \to X$ making the diagram commutative.

"$\Rightarrow$" Suppose $\exists g: Y \to X$ such that $g\circ f = \mathrm{id}_X$. Complete $X \xrightarrow{f} Y$ to a distinguished triangle by (TR1) and compare again
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r, squiggly, "h"] \arrow[d, dashed, "\exists"] & X[1] \arrow[d, equal] \\ X \arrow[r, "\mathrm{id}"] & X \arrow[r] & 0 \arrow[r, squiggly] & X[1] \end{tikzcd}$$
By (TR3) $\exists$ $Z \to 0$ making the diagram commutative $\Rightarrow$ $\mathrm{id}_{X[1]} \circ h = 0 \Rightarrow h = 0$

$\square$

When working with abelian categories, (co)homological functors and long exact (co)homology sequences are very useful.

12.10 Definition: A covariant functor $\mathcal{T} \to \mathcal{A}$ from a triangulated category to an abelian category is called homological (or covariant cohomological) if it transforms distinguished triangles into long exact sequences.
(Similarly for contravariant functors.)

Hom-functors are examples of such (co)homological functors:

12.11 Proposition: Let $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{h}{\rightsquigarrow} X[1]$ be a distinguished triangle in $\mathcal{T}$ and U any object in $\mathcal{T}$. Then applying $\mathrm{Hom}_{\mathcal{T}}(U,-)$ yields a long exact sequence of abelian groups (infinite in both directions)

$$\ldots \to \mathrm{Hom}_{\mathcal{T}}(U, Z[-2]) \to \mathrm{Hom}_{\mathcal{T}}(U, X[-1]) \to \mathrm{Hom}_{\mathcal{T}}(U, Y[-1]) \to \mathrm{Hom}_{\mathcal{T}}(U, Z[-1]) \to$$
$$\to \mathrm{Hom}_{\mathcal{T}}(U, X) \to \mathrm{Hom}_{\mathcal{T}}(U, Y) \to \mathrm{Hom}_{\mathcal{T}}(U, Z) \to \mathrm{Hom}_{\mathcal{T}}(U, X[1]) \to \ldots$$

and applying $\mathrm{Hom}_{\mathcal{T}}(-,U)$ yields a long exact sequence of abelian groups

$$\longrightarrow \mathrm{Hom}_{\mathcal{T}}(Y[1], U) \to \mathrm{Hom}_{\mathcal{T}}(X[1], U) \to \mathrm{Hom}_{\mathcal{T}}(Z, U) \to \mathrm{Hom}_{\mathcal{T}}(Y, U) \to$$
$$\to \mathrm{Hom}_{\mathcal{T}}(X, U) \to \mathrm{Hom}_{\mathcal{T}}(Z[-1], U) \to \ldots$$

Proof (for the covariant case): By 12.9 (a), the composition of two subsequent morphisms in the triangle and hence in the Hom-sequence is zero. Using (TR2) to rotate the triangle, it is sufficient to look at one particular term, for instance at $Z \xrightarrow{h} X[1]$. Let $\alpha: U \to Z$ such that $h \circ \alpha = 0$. We have to find $\beta: U \to Y$ such that $\alpha = g \circ \beta$. The diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, squiggly, "h"] & X[1] \\ 0 \arrow[u, "0"] \arrow[r] & U \arrow[u, dashed, "\exists\beta"] \arrow[r, "id"'] & U \arrow[u, "\alpha"] \arrow[r, squiggly, "0"'] \arrow[ur, phantom, "\circlearrowleft"] & 0 \arrow[u, "0"] \end{tikzcd}$$

commutes $\overset{(TR3)}{\Rightarrow}$ $\exists \beta: U \to Y$ such that $\alpha \circ id = g \circ \beta$ □

This implies an analogue of the 5-lemma:

12.12 Corollary: Let

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d, "f"] & U_2 \arrow[r] \arrow[d, "g"] & U_3 \arrow[r, squiggly] \arrow[d, "h"] & U_1[1] \arrow[d, "f[1]"] \\ V_1 \arrow[r] & V_2 \arrow[r] & V_3 \arrow[r, squiggly] & V_1[1] \end{tikzcd}$$

be a morphism of distinguished triangles.

If two of the morphisms f, g, h are isomorphisms, then the third one is so, too.

In particular, given a morphism $X \xrightarrow{\alpha} Y$ and two distinguished triangles above α: $X \xrightarrow{\alpha} Y \to Z \rightsquigarrow X[1]$

and $X \xrightarrow{\alpha} Y \to Z' \rightsquigarrow X[1]$ then $Z \simeq Z'$!

(Z is often called the <u>cone</u> of α. It is unique up to non-unique isomorphism. In other words, forming the cone is not functorial, unlike kernel or cokernel in abelian categories. There is no universal property of the cone unless one imposes additional assumptions.)

Proof of 12.12: Using
$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "id_X"] & Y \arrow[r] \arrow[d, "id_Y"] & Z \arrow[r, squiggly] & X[1] \arrow[d, "id_{X[1]}"] \\ X \arrow[r, "\alpha"] & Y \arrow[r] & Z' \arrow[r, squiggly] & X[1] \end{tikzcd}$$
, the second statement follows from the first one. why?

By rotating the triangle, it is sufficient to show that f, g isomorphisms implies h is an isomorphism.

Applying the functor $Hom_{\mathcal{T}}(V_3, -)$ to both triangles yields
$$\begin{tikzcd} \cdots \arrow[r] & Hom_{\mathcal{T}}(V_3, U_1) \arrow[r] \arrow[d, "f"] & Hom_{\mathcal{T}}(V_3, U_2) \arrow[r] \arrow[d, "g"] \arrow[l, phantom, "\circlearrowleft"] & Hom_{\mathcal{T}}(V_3, U_3) \arrow[r] \arrow[d, "h"] \arrow[l, phantom, "\circlearrowleft"] & Hom_{\mathcal{T}}(V_3, U_1[1]) \arrow[r] \arrow[d, "f[1]"] & \cdots \\ \cdots \arrow[r] & Hom_{\mathcal{T}}(V_3, V_1) \arrow[r] & Hom_{\mathcal{T}}(V_3, V_2) \arrow[r] & Hom_{\mathcal{T}}(V_3, V_3) \arrow[r] & Hom_{\mathcal{T}}(V_3, V_1[1]) \arrow[r] & \cdots \end{tikzcd}$$

The 5-lemma for abelian categories implies that $h: Hom_{\mathcal{T}}(U_3, U_3) \to Hom_{\mathcal{T}}(V_3, V_3)$ is an isomorphism, too. why? $\Rightarrow id_{V_3}$ is in the image and h has a left inverse.

Using $Hom_{\mathcal{T}}(-, U_3)$ yields a right inverse of h. $\square$

We will not go deeper into the huge theory of triangulated categories, which for us is merely a set of tools and a language. Before we check that homotopy categories are triangulated, we consider another class of examples; recall from chapter 10:

<u>12.13 Definition:</u> Let A be a finite dimensional algebra and A-mod the abelian category of finite dimensional A-modules. The <u>stable category</u> (of left A-modules) $A\text{-}\underline{mod} = A\text{-stmod}$ has objects: the finite dimensional A-modules and morphism sets $\underline{Hom}_A(X,Y) := Hom_A(X,Y) / \{f: X \to Y \text{ factoring } X \xrightarrow{f} Y \text{ through some } P \text{ projective}\}$
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ P \arrow[ur] \end{tikzcd}$$

One can, of course, replace P projective by other kinds of modules. In particular,
$\overline{\mathrm{Hom}}_A(X,Y) := \mathrm{Hom}_A(X,Y) / \{f: X\to Y \text{ factoring } X \xrightarrow{f} Y \text{ through some } I \text{ injective}\}$
defines the injective stable category.
In A-$\underline{\mathrm{mod}}$, P projective is isomorphic to 0, since $id_P = 0 \in \underline{\mathrm{Hom}}_A(P,P)$.

A-$\underline{\mathrm{mod}}$ is an additive category, but usually not abelian. To define a triangulated structure we need to choose an interesting auto-equivalence $\Sigma: A\text{-}\underline{\mathrm{mod}} \to A\text{-}\underline{\mathrm{mod}}$. An interesting operation on modules is the Heller operator

$$\Omega: X \longmapsto \Omega(X) \text{ where } 0\to\Omega(X)\to P\to X\to 0 \text{ is exact for some } P \text{ projective},$$

i.e. $\Omega(X)$ = first syzygy, or its "inverse"

$$\Omega^{-1}: X \to \Omega^{-1}(X) \text{ where } 0\to X\to I\to\Omega^{-1}(X)\to 0 \text{ is exact for some}$$

I injective.
On A-mod, $\Omega(X)$ and $\Omega^{-1}(X)$ are not well-defined, since there is a choice of P or I to be made, respectively. On A-$\underline{\mathrm{mod}}$, $\Omega(X)$ is well-defined and even functorial, i.e. also well-defined on morphisms. Ω^{-1} works on A-$\overline{\mathrm{mod}}$, modulo injectives. But Ω and Ω^{-1} are not auto-equivalences, in general. After all, $\Omega^n(X)$ can be zero for X not projective or injective. So we won't find a triangulated structure on A-$\underline{\mathrm{mod}}$ in general, using $\Sigma = \Omega$.
However, we will find one for A self-injective.
Recall: A self-injective means projective = injective, in particular $_AA$ is both projective and injective and A-$\underline{\mathrm{mod}} = A$-$\overline{\mathrm{mod}}$, which makes the situation symmetric.

12.14 Theorem: Let A be a self-injective algebra. Then A-$\underline{\mathrm{mod}}$ is a triangulated with shift $\Sigma = \Omega^{-1}$.
Proof: Ω^{-1} is a self-equivalence of A-$\underline{\mathrm{mod}}$. We have to define what a distinguished triangle is. We will, of course, construct the distinguished triangles from short exact sequences in A-mod. Let $0\to X\xrightarrow{f} Y\xrightarrow{g} Z\to 0$ be an ses in A-mod. We want a distinguished triangle $X\xrightarrow{f} Y\xrightarrow{g} Z\xrightarrow{h} \Sigma X = \Omega^{-1}(X)$ (where the morphisms represent residue classes in $\underline{\mathrm{Hom}}$). So, we have to define
$h: Z\to\Omega^{-1}(X)$ or $\Sigma^{-1}h: \Omega(Z)\to X$.

12.20

Choose P projective and a surjection $P \to Z$.

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \\
0 \arrow[r] & \Omega(Z) \arrow[r] \arrow[u, "\alpha'"] \arrow[ur, phantom, "\circlearrowleft"] & P \arrow[r] \arrow[u, "\exists \alpha"] \arrow[ur, phantom, "\circlearrowleft"] & Z \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}
\qquad (*)
$$

P projective $\Rightarrow \alpha$ exists, and α restricts to $\alpha' : \Omega(Z) \to X$

In A-$\underline{\text{mod}}$, α' is unique: Suppose both α_1 and α_2 lift id_Z, so $\beta := \alpha_1 - \alpha_2$ lifts 0

$$
\rightsquigarrow
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \\
0 \arrow[r] & \Omega(Z) \arrow[r] \arrow[u, "\beta'"] \arrow[ur, phantom, "\circlearrowleft"] & P \arrow[r] \arrow[u, "\beta"] \arrow[ur, phantom, "\circlearrowleft"] & Z \arrow[r] \arrow[u, "0"] & 0
\end{tikzcd}
$$

By the universal property of the kernel, β' exists, and β' factors through $P \Rightarrow \alpha' : \Omega(Z) \to X$ is well-defined in A-$\underline{\text{mod}}$. Set $h := \Sigma\alpha' : Z \to \Sigma X$.

Now we have defined some distinguished triangles. In general, anything isomorphic to such a triangle is called a distinguished triangle. $\Rightarrow$ The first requirement in (TR1) is satisfied.

The second requirement in (TR1) also is easy: $0 \to X \xrightarrow{id} X \to 0 \to 0$ is a ses.

Let $X \xrightarrow{f} Y$ be any morphism. To find a triangle starting with f, we choose an injective envelope $i : X \to I_X$ and get a ses $0 \to X \xrightarrow{(f,i)} Y \oplus I_X \to \underset{\text{cokernel}}{Z} \to 0$

In A-$\underline{\text{mod}}$: $I_X = 0 \rightsquigarrow X \to Y \to Z \rightsquigarrow \Sigma X$ is a distinguished triangle.

(TR2) Up to isomorphism, we are given a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Omega^{-1}(X)$ coming from an ses $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$. To find a rotated triangle

$$Z[-1] = \Omega(Z) \to X \to Y \to Z$$

we use $(*)$ above, which defines $\alpha' : \Omega(Z) \to X$, and $\alpha' = \Sigma^{-1}h = h[-1]$. The sign can be adjusted as $-id$ is an isomorphism. Rotating in the other direction works similarly.

(TR3) Again up to isomorphism, given are ses $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ and $0 \to X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \to 0$ such that there is a commutative diagram that we need to complete

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] & \Omega^{-1}X \arrow[d, "\Omega^{-1}(\alpha)"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Omega^{-1}X'
\end{tikzcd}
$$

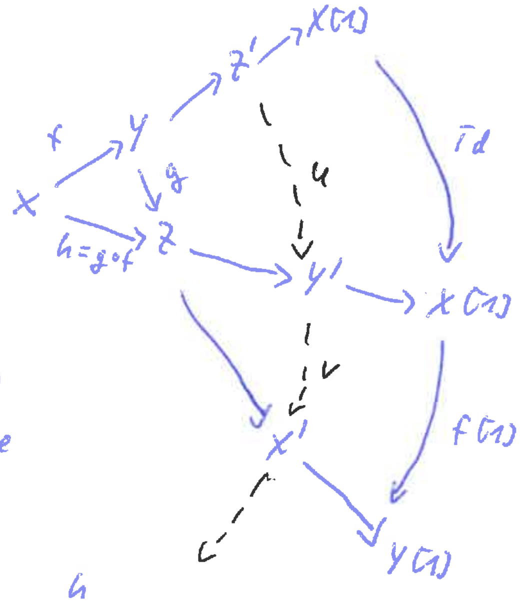
Commutativity happens in the stable category, that is, $\beta \circ f - f' \circ \alpha$ factors through a projective module:

$\Rightarrow \exists\, P$ projective such that
$$\begin{tikzcd} X \arrow[rr, "\beta\circ f - f'\circ\alpha"] \arrow[dr, "s"'] & \arrow[d, phantom, "\circlearrowright"] & Y' \\ & P \arrow[ur, "t"'] & \end{tikzcd}$$
commutes.

Form an ses $0 \to X \xrightarrow{(f,-s)} Y\oplus P \to \tilde{Z} \to 0$ and get a commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "{(f,-s)}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & Y\oplus P \arrow[r] \arrow[d, "\pi_Y"] \arrow[dr, phantom, "\circlearrowright"] & \tilde{Z} \arrow[r] \arrow[d, "\exists\ \text{why?}"] & 0 \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$
(with π_Y the projection onto Y) yielding an isomorphism of triangles in A-$\underline{\mathrm{mod}}$:
$$\begin{tikzcd} X \arrow[r, "{(f,-s)}"] \arrow[d, equal] & Y\oplus P \arrow[r] \arrow[d, "\pi_Y"] & \tilde{Z} \arrow[r] \arrow[d] & \Omega^{-1}(X) \arrow[d, equal] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Omega^{-1}(X) \end{tikzcd}$$

Hence we can replace $X \to Y \to Z \to \Omega^{-1}X$ by $X \to Y\oplus P \to \tilde{Z} \to \Omega^{-1}(X)$, which (in A-$\underline{\mathrm{mod}}$) means replacing the given commuting square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$
by the square
$$\begin{tikzcd} X \arrow[r, "{(f,-s)}"] \arrow[d, "\alpha"'] & Y\oplus P \arrow[d, "{(\beta,t)}"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$
which now really commutes in the module category, by construction. This implies the existence of the desired map $\tilde{Z} \to Z'$ yielding a morphism of triangles.

(TR4) Given
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & \arrow[u, phantom, "\circlearrowright"] & Z \end{tikzcd}$$
we complete it to distinguished triangles



and we have to find u and v and then check the required commutativities.
Idea: use the analogy between the octahedral axiom and the third isomorphism theorem.

The commutative diagram with $\overset{h}{\widetilde{g\circ f}} = g\circ f$ is in A-$\underline{\mathrm{mod}}$. As before we can add projective summands to get commutativity in A-mod. Let us assume we have done that already. Thus in A-mod there are two commutative diagrams:

$$(\ast\ast)\quad \begin{tikzcd} 0 \arrow[r] & X \arrow[r, "g\circ f"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, dashed, "\exists v"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & X' \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, dashed, "\exists u"] & 0 \\ 0 \arrow[r] & X \arrow[r, "h"', "=g\circ f"'] & Z \arrow[r] & Y' \arrow[r] & 0 \end{tikzcd}$$

where X', Y', Z' are cokernels

and u and v exist by the universal property of the cokernel.

Here, $Z' \simeq Y/X$, $Y' \simeq Z/X$, $X' \simeq Z/Y \simeq {}^{Z/X}/_{Y/X} \simeq {}^{Y'}/_{Z'}$ $\Rightarrow \exists$ ses $0 \to Z' \to Y' \to X' \to 0$

$\Rightarrow \exists$ distinguished triangle

$$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \rightsquigarrow$$

It remains to check commutativities.

By construction of v and of u:
$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Z' \arrow[d, "u"] \\ Z \arrow[r] & Y' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z \arrow[r] \arrow[dr] & Y' \arrow[d, "v"] \arrow[dl, phantom, "\circlearrowright"] \\ & X' \end{tikzcd}$$

There are two further diagrams to be checked:

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "v"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y[1] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Omega(Y') = Y'[-1] \arrow[r] \arrow[d, "v[-1]"] \arrow[dr, phantom, "(+)"] & X \arrow[d, "f"] \\ \Omega(X') = X'[-1] \arrow[r] & Y \end{tikzcd}$$

We only check the second one, by going back to the defining ses.

In the definition of v we used the above diagram $(\ast\ast)$, which is commutative in A-mod. We also know that $\Omega(X')$ and $\Omega(Y')$ can be defined to make the following diagrams commutative:
$$\begin{tikzcd} X \arrow[r] & Z \arrow[r] & Y' \\ \Omega(Y') \arrow[r] \arrow[u, "\alpha"] & P \arrow[r] \arrow[u] & Y' \arrow[u, equal] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r] & Z \arrow[r] & X' \\ \Omega(X') \arrow[r] \arrow[u, "\beta"] & Q \arrow[r] \arrow[u] & X' \arrow[u, equal] \end{tikzcd} \quad (P, Q \text{ projective})$$

Put the diagrams together and get

$$\begin{tikzcd} & \Omega(X') \arrow[r] \arrow[d] \arrow[dl, equal] & Q \arrow[r] \arrow[d] & X' \arrow[d, equal] \\ \Omega(X') \arrow[r] & Y \arrow[r] & Z \arrow[r] & X' \\ \Omega(Y') \arrow[r] \arrow[u, "v[-1]"] \arrow[ur, phantom, "(+)"] & X \arrow[r] \arrow[u, "f"] \arrow[ur, phantom, "\hat{v}"] & Z \arrow[r] \arrow[u, equal] \arrow[ur, phantom, "v"] & Y' \arrow[u, "v"] \\ & \Omega(Y') \arrow[r] \arrow[u] \arrow[ul, equal] & P \arrow[r] \arrow[u] \arrow[uuu, bend left] & Y' \arrow[u, equal] \arrow[uuu, bend left] \end{tikzcd}$$

where the blue part commutes.

There exists a lift $\hat{v}$ of v why?

and $v[-1]$ is the restriction. This implies commutativity of $(+)$.

$\square$

This proof did not really use A-modules. Slightly modified it also works for stable categories of Frobenius categories:
An exact category is a full and extension closed subcategory of an abelian category. It is called a <u>Frobenius category</u> when it has enough projectives and injectives and projective objects coincide with injective objects. Factoring out morphisms that factor through projectives defines a stable category which carries the same kind of triangulated structure as $A-\underline{\text{mod}}$ (which is a special case).

Now we turn to the second class of examples, homotopy categories. We are interested in $K(A\text{-Mod})$ or $K(A\text{-mod})$ or similar categories. The triangulated structure is present in a much more general setup: When $\mathcal{A}$ is an additive category, one can define (co)chain complexes, $Ch(\mathcal{A})$, homotopy and $K(\mathcal{A})$. One also can define left bounded, right bounded and bounded (on both sides) complexes (requiring appropriate terms to vanish) and categories $Ch^+(\mathcal{A})$, $Ch^-(\mathcal{A})$, $Ch^b(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$. When $\mathcal{A}$ is abelian, (co)homology is defined and one also can define for instance $Ch^{-,b}(\mathcal{A})$, where the first superscript bounds the complexes and the second one bounds the (co)homology.

<u>12.15 Theorem:</u> Let $\mathcal{A}$ be an additive category. Then the homotopy category $K(\mathcal{A})$ is triangulated, and so are its subcategories $K^{x,y}(\mathcal{A})$ for $x,y \in \{\emptyset,+,-,b\}$ whenever they are defined.
Proof: We have already (on page 12.5) a shift, which we now take as auto-equivalence of $Ch(\mathcal{A})$: $X[1]$ has $X[1]_n = X_{n+1}$. This also works for all the subcategories we are interested in, and of course also for the homotopy categories.
We define particular distinguished triangles and then call all triangles distinguished that are isomorphic to these.
Let $\alpha: X^\cdot \to Y^\cdot$ be a morphism of complexes. Then the mapping cone, defined in 12.2), is the complex $\mathrm{Cone}(f)$ whose n-th term is $X^{n+1} \oplus Y^n$, with differential

$$d = \begin{pmatrix} -d_X & 0 \\ \alpha[1] & d_Y \end{pmatrix} \quad \text{where } -d_X = d_{X[1]} \text{ by definition.}$$

This gives morphisms of complexes:

$$\begin{tikzcd}
X^\bullet \arrow[d, "\alpha"] & \cdots \arrow[r] & X^{n-1} \arrow[r, "d"] \arrow[d, "\alpha"] & X^n \arrow[r, "d"] \arrow[d, "\alpha"] & X^{n+1} \arrow[r, "d"] \arrow[d, "\alpha"] & \cdots \\
Y^\bullet \arrow[d, "\beta"] & \cdots \arrow[r] & Y^{n-1} \arrow[r, "d"] \arrow[d] & Y^n \arrow[r, "d"] \arrow[d] & Y^{n+1} \arrow[r, "d"] \arrow[d] & \cdots \\
C(\alpha) \arrow[d, "\kappa"] & \cdots \arrow[r] & X^n \oplus Y^{n-1} \arrow[r, "d"] \arrow[d] & X^{n+1} \oplus Y^n \arrow[r, "d"] \arrow[d] & X^{n+2} \oplus Y^{n+1} \arrow[r, "d"] \arrow[d] & \cdots \\
X^\bullet[1] & \cdots \arrow[r] & X^n \arrow[r, "d"] & X^{n+1} \arrow[r, "d"] & X^{n+2} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

(α is given, β and κ we know from page 12.5, where we have seen the ses of complexes $Y^\bullet \xrightarrow{\beta} C(\alpha) \xrightarrow{\kappa} X^\bullet[1]$ yielding a long exact cohomology sequence)

Now we check the axioms:

(TR1) By definition, the class of distinguished triangles is closed under isomorphism. Each morphism $\alpha: X^\bullet \to Y^\bullet$ occurs in a distinguished triangle by construction. By 12.4, the cone of id_X is acyclic, and we have checked that it is split exact, which means, $id_{Cone(id)}$ is null homotopic, so $cone(id) \simeq 0 \Rightarrow$ There exists a distinguished triangle $X^\bullet \xrightarrow{id} X^\bullet \to 0^\bullet \rightsquigarrow$.

(TR2) Given a distinguished triangle $X \xrightarrow{\alpha} Y \to cone(\alpha) \to X[1]$, we check that $Y \xrightarrow{i} cone(\alpha) \to X[1] \to Y[1]$ is distinguished, too. (The other case is similar.) Here, $i: Y \to cone(\alpha)$ is inclusion. In the following we abbreviate $cone(\alpha)$ by $C(\alpha)$, etc.

There is a distinguished triangle $Y \xrightarrow{i} C(\alpha) \to C(i) \to Y[1]$ and we have to identify $C(i)$ with $X[1]$ such that

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, equal] & C(\alpha) \arrow[r] \arrow[d, equal] & C(i) \arrow[r] & Y[1] \arrow[d, equal] \\
Y \arrow[r, "i"] & C(\alpha) \arrow[r] & X[1] \arrow[r, "{\alpha[1]}"] \arrow[u] & Y[1]
\end{tikzcd}$$

is commutative.

Write $C(i)$ explicitly:

$$\begin{tikzcd}
Y \arrow[d, "i"] & \cdots \arrow[r] & Y^{n-1} \arrow[r] \arrow[d] & Y^n \arrow[r] \arrow[d] & Y^{n+1} \arrow[r] \arrow[d] & \cdots \\
C(\alpha) \arrow[d] & \cdots \arrow[r] & X^n \oplus Y^{n-1} \arrow[r] & X^{n+1} \oplus Y^n \arrow[r] & X^{n+2} \oplus Y^{n+1} \arrow[r] & \cdots \\
\text{Cone } C(i) & \cdots \arrow[r] & \underbrace{Y^n}_{Y[1]} \oplus \underbrace{X^n \oplus Y^{n-1}}_{C(\alpha)} \arrow[r] & Y^{n+1} \oplus X^{n+1} \oplus Y^n \arrow[r] & Y^{n+2} \oplus X^{n+2} \oplus Y^{n+1} &
\end{tikzcd}$$

There is an obvious map $C(i) \xrightarrow{\beta} X[1]$, projecting onto the $X^\bullet$-terms. Since i uses the identity on $Y^\bullet$-terms, the kernel of β is the cone of the identity on $Y^\bullet$ check this which is isomorphic to zero, as we know $\Rightarrow C(i) \simeq X[1]$ and the above diagram commutes with this isomorphism.

(The description of $C(i)$ is known as a mapping cylindre. The point here is that the identity on Y^n occurs in the differential of this mapping cylindre and therefore the cone of id_Y is present.)

(TR3) Given

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d, "g"] & C(\alpha) \arrow[r] \arrow[d, dashed, "h?"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "\beta"] & Y' \arrow[r] & C(\beta) \arrow[r] & X'[1]
\end{tikzcd}$$

we need to find h to complete the diagram into a commutative diagram, which is a morphism of triangles.

$C(\alpha)$ has terms $X^n \oplus Y^{n-1}$ and $C(\beta)$ has terms $(X')^n \oplus (Y')^{n-1}$ $\Rightarrow$ there is an obvious candidate for h:

$$\begin{matrix} X^n & \xrightarrow{f^n} & (X')^n \\ \oplus & & \oplus \\ Y^{n-1} & \xrightarrow[g^{n-1}]{} & (Y')^{n-1} \end{matrix}$$

Since the maps to and from the cones are just inclusions/projection onto direct summands, the square commutes.

But one has to check that h is a map of complexes. This follows from

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} -d & 0 \\ \alpha & d \end{pmatrix} = \begin{pmatrix} -d & 0 \\ \beta & d \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$$

since f, g commute with differentials and by assumption, $\beta \circ f = g \circ \alpha$.

(TR4) As before we may assume that the given situation

$$\begin{tikzcd}
 & Y \arrow[d, "g"] \\
X \arrow[ur, "f"] \arrow[r, "h"'] & Z
\end{tikzcd}$$

is already commutative in $Ch(\mathcal{A})$, since subtracting the difference just means choosing another representative.

We complete to standard triangles, where the third terms are cones:

$$\begin{tikzcd}
 & & & X[1] \arrow[dd, bend left, "\mathrm{id}_{X[1]}"] \\
 & Y \arrow[r] \arrow[d, "g"] & C(f) \arrow[ur] \arrow[d, dashed, "u?"] & \\
X \arrow[ur, "f"] \arrow[r, "h"'] & Z \arrow[r] \arrow[d] & C(h) \arrow[r] \arrow[dl, dashed, "v?"] & X[1] \arrow[dl, "f[1]"] \\
 & C(g) \arrow[r] & Y[1] &
\end{tikzcd}$$

Writing down the cones will suggest how to define u and v:

$C(f)$: $X^{n+1} \oplus Y^n$, $\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}$

$C(g)$: $Y^{n+1} \oplus Z^n$, $\begin{pmatrix} -d & 0 \\ g & d \end{pmatrix}$

$C(h)$: $X^{n+1} \oplus Z^n$, $\begin{pmatrix} -d & 0 \\ h & d \end{pmatrix}$

We want: $C(f) \xrightarrow{u} C(h) \xrightarrow{v} C(g)$

$\rightsquigarrow$ define u:

$$\begin{matrix} X^{n+1} & \xrightarrow{1} & X^{n+1} \\ \oplus & & \oplus \\ Y^n & \xrightarrow[g]{} & Z^n \end{matrix}$$

(then v:

$$\begin{matrix} X^{n+1} & \xrightarrow{f} & Y^{n+1} \\ \oplus & & \oplus \\ Z^n & \xrightarrow[1]{} & Z^n \end{matrix}$$

is automatic)

What is left to be done is: • check u and v are morphisms of complexes

- u and v define a distinguished triangle, ie $C(g) \simeq C(u)$
- diagrams commute as required

For instance: $u : C(f) \to C(h)$ is a morphism of complexes, since

$$\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} = \begin{pmatrix} -d & 0 \\ h & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \text{ because } h = g \circ f \text{ and } g \circ d = d \circ g$$

The cone of u is

$$\begin{matrix} X^{n+2} \\ \oplus \\ Y^{n+1} \\ \oplus \\ X^{n+1} \\ \oplus \\ Z^{n} \end{matrix} \xrightarrow{\ d\ } \begin{matrix} X^{n+3} \\ \oplus \\ Y^{n+2} \\ \oplus \\ X^{n+2} \\ \oplus \\ Z^{n+1} \end{matrix}$$

where the differential is

$$\begin{pmatrix} d & 0 & 0 & 0 \\ -f & -d & 0 & 0 \\ 1 & 0 & -d & 0 \\ 0 & g & h & d \end{pmatrix} = d$$

The entry $1 : X^{n+2} \longrightarrow X^{n+2}$ identifies $\operatorname{Cone}(X^\bullet) = C(id_X)$ as a direct summand of $C(u)$, which is homotopic to zero. Hence, we can split off this summand and forget about it. This removes all $X^\bullet$-terms and what is left is exactly $C(g)$.

$\Rightarrow$ In $K(\mathcal{A})$: $C(g) \simeq C(u)$.

Check that the resulting map is v and everything commutes as desired. $\square$