§12. Derived categories

Let R be a ring. The R-modules form an abelian category R-Mod.
We also can form chain or cochain complexes and represent modules by projective or injective resolutions. Complexes form abelian categories, too, which we can modify to obtain derived categories – which are not, in general, abelian categories.

Recall from §6: A chain complex is a sequence of R-modules C_n $(n \in \mathbb{Z})$ and R-module homomorphisms $C_*: \ldots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \xrightarrow{d_{-2}} \ldots$ such that $d_{n-1} \circ d_n = 0$ $\forall n$
$Z_n := \operatorname{Ker}(d_n)$, $B_n := \operatorname{Im}(d_{n+1})$, $H_n(C_*) = H_n := Z_n/B_n$ is the n-th homology of C_*.
A cochain complex is a sequence of R-modules C^n $(n \in \mathbb{Z})$ and R-module homomorphisms
$C^*: \ldots \to C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \ldots$ such that $d^{n+1} \circ d^n = 0$ $\forall n$
$Z^n := \operatorname{Ker}(d^n)$, $B^n := \operatorname{Im}(d^{n-1})$, $H^n(C^*) = H^n := Z^n/B^n$ is the n-th cohomology of C^*.
Instead of R-Mod we may use any abelian category $\mathcal{A}$ to form complexes

We will now use cochain complexes, but everything would work with chain complexes, too.
A morphism between cochain complexes is a commutative diagram of morphisms

$$\begin{tikzcd}
C^* \arrow[d, "f^*"] & \ldots \arrow[r] & C^{-1} \arrow[r, "d"] \arrow[d, "f^{-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & C^0 \arrow[r, "d"] \arrow[d, "f^0"'] \arrow[dr, phantom, "\circlearrowleft"] & C^1 \arrow[r] \arrow[d, "f^1"'] & \ldots \\
D^* & \ldots \arrow[r] & D^{-1} \arrow[r, "d"'] & D^0 \arrow[r, "d"'] & D^1 \arrow[r] & \ldots
\end{tikzcd}$$

⇝ There is a category $Ch(\mathcal{A})$ or $Ch(R\text{-Mod})$ of cochain complexes over $\mathcal{A}$ or over R-Mod.
A morphism $f^*: C^* \to D^*$ is a quasi-isomorphism (q.i.) :⇔ it induces isomorphisms on cohomology. For instance, a module M seen as a complex $\ldots \to 0 \to M \to 0 \to \ldots$ is quasi-isomorphic to a projective resolution and to an injective resolution.
Usually, there is no map of complexes in the opposite direction that would provide an inverse on cohomology.

12.1 Proposition: Ch(R-Mod) and generally Ch($\mathcal{A}$) for $\mathcal{A}$ abelian, are abelian categories. (This is 6.4)

Proof: Let $f^*: C^* \to D^*$ be a morphism of cochain complexes. To define $\mathrm{Ker}(f^*)$ consider

$$\begin{tikzcd}
K^*: \ldots \arrow[r] & K^{-1} \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & \ldots \\
C^*: \ldots \arrow[r] & C^{-1} \arrow[r, "d"] \arrow[d, "f^{-1}"'] & C^0 \arrow[r, "d"] \arrow[d, "f^0"'] & C^1 \arrow[r] \arrow[d, "f^1"'] & \ldots \\
D^*: \ldots \arrow[r] & D^{-1} \arrow[r, "d"] & D^0 \arrow[r, "d"] & D^1 \arrow[r] & \ldots
\end{tikzcd}$$

where $K^{-1} \hookrightarrow C^{-1}$ is the kernel of f^{-1} etc. The differentials on K^* are obtained by requiring that $K^* \to C^*$ is a morphism of complexes, and then applying property of kernels. Cokernels are defined analogously. All axioms then follow from R-Mod or $\mathcal{A}$ being abelian categories. □

This abelian structure implies that a sequence $0 \to C^* \xrightarrow{f} D^* \xrightarrow{g} E^* \to 0$ of complexes is exact if and only if for each $n \in \mathbb{Z}$, the sequence $0 \to C^n \to D^n \to E^n \to 0$ is exact. By Theorem 6.5, such a short exact sequence of cochain complexes induces a long exact cohomology sequence:

$$\begin{tikzcd}
H^*(C^*) \arrow[rr, "H^*(f)"] & & H^*(D^*) \arrow[dl] \\
& H^*(E^*) \arrow[ul, bend left, "\partial"] \arrow[u, phantom, "\partial"] &
\end{tikzcd}$$

A morphism between modules can be lifted to a morphism between projective or injective resolutions, which in turn induces the same morphism on (co)homology. This is a consequence of the Comparison Theorem 6.13, which asserts that a lift of a module homomorphism is unique up to (co-)chain homotopy equivalence. By Lemma 6.11, two homotopic maps between complexes induce the same map on (co)homology.

Recall Definition 6.10: $f: C^* \to D^*$ is null-homotopic $:\Leftrightarrow \exists s^n: C^n \to D^{n-1}$ such that $f = s \circ d + d \circ s$

$$\begin{tikzcd}
\ldots \arrow[r] & C^{n-1} \arrow[r, "d"] \arrow[d, "f^{n-1}"'] & C^n \arrow[r, "d"] \arrow[d, "f^n"'] \arrow[dl, "s^n"'] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] \arrow[dl, "s^{n+1}"'] & \ldots \\
\ldots \arrow[r] & D^{n-1} \arrow[r, "d"'] & D^n \arrow[r, "d"'] & D^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

$f, g: C^* \to D^*$ are homotopic $:\Leftrightarrow$ $g-h$ is null homotopic (by a homotopy s)
$f: C^* \to D^*$ is a homotopy equivalence $\Leftrightarrow$ $\exists\, h: D^* \to C^*$ such that $f \circ h$ is homotopic to 1_{D^*} and $h \circ f$ is homotopic to 1_{C^*}.

We have seen that Definition 6.12 makes sense: The homotopy category $K(R\text{-Mod})$ or $K(\mathcal{A})$ of cochain complexes has as objects the objects in $Ch(R\text{-Mod})$ or $Ch(\mathcal{A})$, respectively, and as morphisms the morphisms between cochain complexes modulo being homotopic.
The homotopy category is an additive category, but in general not an abelian category.
The functor $Ch(R\text{-Mod}) \longrightarrow K(R\text{-Mod})$, sending C^* to itself and f to its residue class, is additive.

The complex $\ldots 0 \to R \xrightarrow{id} R \to 0 \ldots$ has trivial cohomology and the zero map to the zero complex is a quasi-isomorphism.

$$\begin{tikzcd} \ldots 0 \arrow[r] & 0 \arrow[r] \arrow[d, "0"'] & R \arrow[r, "id"] \arrow[d, phantom, "2"] & R \arrow[r] \arrow[d, "0"] & 0 \ldots \\ \ldots 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \ldots & \end{tikzcd}$$

The identity morphism

$$\begin{tikzcd} \ldots 0 \arrow[r] & R \arrow[r, "id"] \arrow[d, "id"'] \arrow[dl, "0"'] & R \arrow[r] \arrow[d, "id"] \arrow[dl, "id"'] & 0 \ldots \arrow[dl, "0"] \\ \ldots 0 \arrow[r] & R \arrow[r, "id"] & R \arrow[r] & 0 \ldots \end{tikzcd}$$

on $\ldots 0 \to R \xrightarrow{id} R \to 0$ is null-homotopic, so id and 0 are homotopic.

This illustrates that in $K(R\text{-Mod})$ it does not make sense to use injectivity or surjectivity of representatives of cosets of morphisms.
Kernels and cokernels usually do not exist in homotopy categories. But there is a kind of analogue:

12.2 Definition: Let $f: B^* \to C^*$ be a morphism of cochain complexes. The **mapping cone** of f is the complex $\mathrm{Cone}(f)$ whose n-th term is $B^{n+1} \oplus C^n$ and differential $d^n: B^{n+1} \oplus C^n \longrightarrow B^{n+2} \oplus C^{n+1}$

$$(b, c) \longmapsto (-d_B(b), -f(b) + d_C(c))$$

$$d: \begin{tikzcd} B^{n+1} \arrow[r, "-d"] \arrow[dr, "-f"] & B^{n+2} \\ \oplus & \oplus \\ C^n \arrow[r, "d"] & C^{n+1} \end{tikzcd}$$

(or with column vector: $\begin{pmatrix} b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} -d_B(b) \\ -f(b) + d_C(c) \end{pmatrix} = \begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}$),

Squaring the matrix shows $(d)^2 = 0$ and $\mathrm{Cone}(f)$ is a cochain complex.

As an example, we consider the cone of the identity morphism $id: B^* \to C^* = B^*$, for some complex B^*. Here, $f = id \Rightarrow d = \begin{pmatrix} -d_B & 0 \\ -id & d_C \end{pmatrix}$

Claim: $\exists s: B^n \longrightarrow B^{n-1}$ such that

$d \circ s \circ d = d$. Moreover, B^* is exact.

We first check that B^* is exact, that is, its cohomology vanishes in all degrees. (Such complexes often are called <u>acyclic</u> complexes.)

To show: $\mathrm{Im}(d) = \mathrm{Ker}(d)$. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{Ker}(d) \Rightarrow \begin{pmatrix} -d_B(x) \\ -f(x) + d_C(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -x = -\underset{id}{\underset{\shortparallel}{f}}(x) = d_{\underset{B}{\underset{\shortparallel}{C}}}(y)$ (and then $d(x) = d^2(y) = 0$)

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d(y) \\ 0 \end{pmatrix} = d\left(\begin{pmatrix} -y \\ 0 \end{pmatrix}\right) \in \mathrm{Im}(d) \Rightarrow \text{Cone}(id) \text{ is acyclic.}$$

Now set $s: B^{n+2} \oplus \overset{B^{n+1}}{\overset{\shortparallel}{C^{n+1}}} \longrightarrow B^{n+1} \oplus \overset{B^n}{\overset{\shortparallel}{C^n}}$ by $s: \overset{\in B^{n+2} \oplus B^{n+1}}{\begin{pmatrix} x \\ y \end{pmatrix}} \longmapsto \overset{\in B^n \oplus B^{n+1}}{\begin{pmatrix} 0 \\ -x \end{pmatrix}}$, ie s given by $\begin{pmatrix} 0 & -id \\ 0 & 0 \end{pmatrix}$

Compute $d \circ s \circ d = d$ in terms of matrices:

$$\begin{pmatrix} -d & 0 \\ -\underset{id}{\underset{\shortparallel}{f}} & d \end{pmatrix} \begin{pmatrix} 0 & -id \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d & 0 \\ -\underset{id}{\underset{\shortparallel}{f}} & d \end{pmatrix} = \begin{pmatrix} 0 & d \\ 0 & id \end{pmatrix} \begin{pmatrix} -d & 0 \\ -id & d \end{pmatrix} = \begin{pmatrix} -d & 0 \\ -\underset{f}{\underset{\shortparallel}{id}} & d \end{pmatrix} \checkmark$$

$d \circ s \circ d = d$ implies that $d \circ s$ (and $s \circ d$) is an idempotent and C^n can be decomposed as $C^n = Z^n \oplus$ complement and $\underset{\underset{\text{Kernel}(s \circ d)}{\cup}}{\underset{\text{Kernel}(d)}{\underset{\cup}{Z^n}}} = \underset{\underset{d \circ s \circ d}{\shortparallel}}{\underset{\text{Image}(d)}{\underset{\cup}{B^n}}} \oplus \underset{0}{\underset{\cup}{\text{complement}}}$ since $C^* = B^*$ is exact

Why?

This explains why complexes with such a property ($\exists s: d \circ s \circ d = d$) are called <u>split</u>. When the complex is split and in addition exact it is called <u>split exact</u>.

This is stronger than just exact (= acyclic), for instance

$$\_\_ \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \sim \quad \text{over } R = \mathbb{Z}$$

$$\text{or } \_ \xrightarrow{\cdot x} k[x]/x^2 \xrightarrow{\cdot x} k[x]/x^2 \xrightarrow{\cdot x} \sim \quad \text{over } R = k[x].$$

A (co)chain complex C is split exact $\Leftrightarrow id_C$ is null homotopic:

If $\exists s: id_C = d \circ s + s \circ d$, then $d = d \circ id_C = d \circ d \circ s + d \circ s \circ d = d \circ s \circ d$, and conversely decomposing the terms in a split exact complex tells us how to define the homotopy.

$\text{Cone}(id_B) =: \text{Cone}(B)$. $\overset{C}{\overset{\cup}{B}}$ is a subcomplex of $\text{Cone}(B)$.

What kind of information is contained in Cone(f)?

Cone(f) is defined for $f: B \to C$. There always is a map $C \to \text{Cone}(f)$ given by $\underset{\overset{\cap}{C^n}}{c} \longmapsto \underset{\overset{\cap}{B^{n+1} \oplus C_n}}{\binom{0}{c}}$ and this is an injective map of complexes. For $f \neq 0$, one cannot map B in the same way into Cone(f). But one can map Cone(f) to a modified version of B: For any p, define the p-th translate $B[p]$ of B by

$B[p]_n := B_{p+n}$ (so: B_p is the degree 0 part of $B[p]$)

and differential $(-1)^p d$ (we will see that the sign is convenient)

Now there is a map $\delta: \text{Cone}(f) \to B[+1]$, which is a map of complexes:

$$\begin{tikzcd}
\binom{b}{c} \arrow[r, mapsto] \arrow[dd, mapsto] & -b \arrow[dr, mapsto] & \\
& & -d(-b) \arrow[d, equal] \\
\begin{pmatrix} -d(b) \\ -f(b)+d(c) \end{pmatrix} \arrow[rr, mapsto] & & d(b)
\end{tikzcd}$$

This yields a short exact sequence $0 \to C \to \underset{\overset{\uparrow}{\text{in degree } n:\ B^{n+1}}}{\text{Cone}(f)} \xrightarrow{\delta} \underset{\overset{\uparrow}{\text{in degree } n:\ B^{n+1}}}{\overset{\overset{\text{for chain complexes, here } B[-1] \text{ would appear}}{\downarrow}}{B[+1]}} \to 0$

which has a long exact cohomology sequence: $\dots \to H^n(C) \to H^n(\text{Cone } f) \xrightarrow{\delta} \underbrace{H^n(B[1])}_{\overset{\shortparallel}{H^{n+1}(B)}} \xrightarrow{\partial} H^{n+1}(C) \to \dots$

(∂ is the connecting homomorphism).

What is the map $\partial: H^{n+1}(B) \to H^{n+1}(C)$?

Claim: $\partial(\underset{\overset{\shortparallel}{\text{residue class of } b, \text{ a cycle}}}{[b]}) = [f(b)]$. Proof: δ is surjective, b has preimage $\binom{-b}{0}$ in Cone(f). Apply the differential: $d: \binom{-b}{0} \longmapsto \binom{d(-b)}{-f(-b)} = \binom{0}{f(b)}$, since b is a cycle.

Check in chapter 6 that this proves the claim.

Sometimes, for instance when $B = C$ and $f = \text{id}$, B is a subcomplex of Cone(f). In general, however, B is not a subcomplex of Cone(f). Then one may ask: Is there a morphism of complexes $\hat{f}: \text{Cone}(B) \to C$ such that on elements in the terms of B the map $\hat{f}$ is given by f: $\hat{f}(b) = f(b)$? In other words, can one extend $f: B \to C$ to $\hat{f}: \text{Cone}(B) \to C$? A positive answer is equivalent to a property of f:

$$\begin{tikzcd}
\text{Cone}(B) \arrow[r, "\hat{f}"] & C \\
B \arrow[u, phantom, "\cup"] \arrow[ur, "f"']
\end{tikzcd}$$

12.3 Proposition: A morphism $f: B \to C$ of cochain complexes can be extended to $\hat{f}: \mathrm{Cone}(B) \to C \iff f$ is null homotopic.

Proof: Let f be null homotopic by s such that $f = d\circ s + s\circ d$.
Define $\hat{f}: \mathrm{Cone}(B) \to C$ by $\begin{pmatrix}x\\y\end{pmatrix} \longmapsto f(y) - s(x)$. This extends $f: y \mapsto f(y)$.
We have to show that $\hat{f}$ is a morphism of complexes.

$$\begin{tikzcd}
\begin{pmatrix}x\\y\end{pmatrix} \arrow[r, mapsto, "\hat{f}"] \arrow[d, mapsto, "d"'] & f(y)-s(x) \arrow[dr, mapsto, "d"] & \\
 & & d(f(y))-d(s(x)) \arrow[d, phantom, "\| \checkmark"] \\
\begin{pmatrix}-d(x)\\ -\underset{\underset{rd(x)}{u}}{x}+d(y)\end{pmatrix} \arrow[r, mapsto, "\hat{f}"] & f(-x+d(y))-s(-d(x)) & =-f(x)+\underbrace{f(d(y))}_{=d(f(y))}+\underbrace{s(d(x))}_{=f(x)-d(s(x))}
\end{tikzcd}$$

Conversely, let $\hat{f}$ be an extension of f. The first part of the proof suggests to set $s(x) := -\hat{f}(\begin{pmatrix}x\\0\end{pmatrix})$. We have to show: $f = d\circ s + s\circ d$.

$$(d\circ s+s\circ d)(x) = d(-\hat{f}\begin{pmatrix}x\\0\end{pmatrix}) - \underbrace{\hat{f}(d(x))}_{=\hat{f}\begin{pmatrix}d(x)\\0\end{pmatrix}} = \underbrace{-\hat{f}(d\begin{pmatrix}x\\0\end{pmatrix})}_{-\hat{f}\begin{pmatrix}-d(x)\\-x\end{pmatrix}} - \hat{f}\begin{pmatrix}d(x)\\0\end{pmatrix}$$

$$= -\hat{f}\begin{pmatrix}0\\-x\end{pmatrix} = f(x) \checkmark \qquad \square$$

For us it will be important to know if f is a quasi-isomorphism. This is equivalent to a property of $\mathrm{Cone}(f)$:

12.4 Proposition: A morphism $f: B \to C$ of cochain complexes is a quasi-isomorphism $\iff$ $\mathrm{Cone}(f)$ is acyclic.

Proof: As above we consider the short exact sequence of complexes

$$0 \to C \xrightarrow{\alpha} \mathrm{Cone}(f) \xrightarrow{\beta} B[1] \to 0$$

$$\alpha: c \longmapsto \begin{pmatrix}0\\c\end{pmatrix} \qquad \beta: \begin{pmatrix}b\\c\end{pmatrix} \longmapsto -b$$

This comes with the long exact cohomology sequence

$$\dots \to H^n(C) \to H^n(\mathrm{Cone} f) \to H^{n+1}(B) \xrightarrow{\partial} H^{n+1}(C) \to \dots$$

$\mathrm{Cone}(f)$ is exact = acyclic if and only if $H^n(\mathrm{Cone} f) = 0$ $\forall n$ if and only ∂ is an isomorphism $\forall n$. As we have seen above the connecting homomorphisms coincide with $H^n(f)$. $\Rightarrow$ $\mathrm{Cone}(f)$ acyclic is equivalent to all $H^n(f)$ being isomorphisms. $\square$

When B is an acyclic complex, the zero map $0: B \to 0$, from B to the zero complex $\_ \to 0 \to 0 \to \_$ is a qis and the zero map $0: 0 \to B$ is a qis, too.

When $M \neq 0$ is a module with projective resolution P_* and injective resolution I^*, then there is a qis of chain complexes $P_* \to M$ ($= \_ \to 0 \to M \to 0 \to \_$) and a qis of cochain complexes $M \to I^*$. But usually there is no chain map $M \to P_*$ and no cochain map $I^* \to M$. Examples?

So, there are morphisms that are kind of isomorphisms, but do not have inverses.

On the other hand, all projective resolutions of M are (by the Comparison Theorem 6.13) homotopy equivalent to each other, i.e. isomorphic in the homotopy category of chain complexes. And similarly for injective resolutions.

$\rightsquigarrow$ There appear to be two deficiencies in the concept of homotopy category:

- Inverses to quasi-isomorphisms are missing.
- We don't know how to compare projective and injective resolutions, which do not even naturally live in the same homotopy category.

The first problem is the more important one, and we know already how to solve an analogous problem:

When $R = \mathbb{Z}$ or R is any commutative integral domain, we can make all non-zero elements invertible by forming the field of fractions $\mathrm{Quot}(R)$ or $\mathrm{Frac}(R)$: Its elements are equivalence classes of pairs (a,b) with $b \neq 0$ (write: $\frac{a}{b}$) where $\frac{a}{b} \sim \frac{c}{d} \Leftrightarrow ad = bc$.

Then $R \longrightarrow \mathrm{Frac}(R)$, $r \longmapsto \frac{r}{1}$ is an injective maps of rings.

Example: $\mathbb{Q} = \mathrm{Frac}(\mathbb{Z})$.

More generally one can form fractions with denominators in a multiplicative system S, i.e. $S \subset R$, $1 \in S$ and $a, b \in S \Rightarrow a \cdot b \in S$. $S = R \setminus \{0\}$ is a multiplicative system when R has no zero divisors. Example: $R = \mathbb{Z}$, $S = \{n \in \mathbb{Z} : 2 \text{ does not divide } n\}$. Then the fractions $\frac{a}{b}$, $a \in R$, $b \in S$ form a ring in which 2 has no inverse, but all other prime numbers p have inverses: $\frac{1}{p}$. Notation: $R[S^{-1}]$ is the resulting ring of fractions, which can be handled by calculus of fractions, like $\mathbb{Q}$.

In commutative algebra the process of forming $R[S^{-1}]$ from R is called localisation. To explain the name think of functions on a variety, which form a ring. Near a point p, "locally" one considers $R[S^{-1}]$ with $S = \{\text{functions } f \text{ with } f(p) \neq 0\}$ to understand what happens near p.

In algebraic number theory or in commutative algebra one often uses $S = R - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. The definition of prime ideal guarantees that S is a multiplicative system. In this way one can compare global properties with respect to all prime ideals with local properties in one or all $R[S^{-1}]$. One can also localize modules M to $R[S^{-1}] \otimes_R M$.

When S contains 0 then $R[S^{-1}] = \{0\}$, which for us is not a ring.
When R is not commutative there can be left fractions ab^{-1} or right fractions $b^{-1}a$ which may differ. (The terminology left or right fraction is not uniform in the literature). In order to multiply $(ab^{-1})\cdot(c\cdot d^{-1})$ one needs an additional condition: $\exists e, f$ such that $b^{-1}c = ef^{-1}$ (ie $be = cf$), then $(ab^{-1})\cdot(c\cdot d^{-1}) = aef^{-1}d^{-1}$. This is an assumption on R. Noncommutative calculus of fractions often is possible, but not always. Under reasonable assumptions it is possible in categories (more precisely: for morphisms) and this is what we are going to do: We take the quasi-isomorphisms as a multiplicative system and make them invertible by localisation. This will now be carried out in a formal way:

12.5 Definition: Let $\mathcal{C}$ be a category and S a class of morphisms. A localisation of $\mathcal{C}$ with respect to S is a category $\mathcal{D} = S^{-1}\mathcal{C}$ with a functor $Q: \mathcal{C} \to S^{-1}\mathcal{C}$ such that (1) and (2) hold true:

(1) $\forall s \in S$: $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$

(2) For all functors $F: \mathcal{C} \to \mathcal{C}'$ such that $F(s)$ is an isomorphism $\forall s \in S$, there is a unique functor $\hat{F}: \mathcal{C}' \to S^{-1}\mathcal{C}$ such that $F = \hat{F} \circ Q$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C}' \\
S^{-1}\mathcal{C} \arrow[ur, dashed, "\exists!\hat{F}"', "\circlearrowleft"] &
\end{tikzcd}$$

The localisation $S^{-1}\mathcal{C}$ may not exist. If it does exist, it is unique up to natural equivalence, because of the universal property (2).
To show existence we will use calculus of fractions for categories, which goes back to Gabriel and Zisman's book "Calculus of fractions and homotopy theory".

12.6 Definition: A class S of morphisms in $\mathcal{C}$ is called a multiplicative system in $\mathcal{C}$:⟺

(1) S is closed under multiplication: $\forall s,t \in S$: $s\circ t \in S$ when it is defined, and $S \ni id_X$ $\forall X$,

(2) (Ore condition) Let $t: Z \to Y$ be in S. Then for each $g: X \to Y$ in $\mathcal{C}$ there exists $s \in S$ and a commutative diagram for some h in $\mathcal{C}$,

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "s"'] & Z \arrow[d, "t"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

Moreover, the symmetric statement (given s there exists t) is valid, too.

(3) (Cancellation rule) Let $f, g: X \to Y$ be in $\mathcal{C}$. Then the following conditions are equivalent: (a) $\exists s \in S$ such that $s\circ f = s\circ g$

(b) $\exists t \in S$ such that $f\circ t = g\circ t$

Example: Let $\mathbb{Z}$ be the category with one object $*$ and $Hom_{\mathbb{Z}}(*,*) = \mathbb{Z}$ with composition by multiplication. Then $S := \mathbb{Z} \setminus \{0\}$ is a multiplicative system and the rational numbers satisfy the condition in 12.5 on the localisation $S^{-1}\mathbb{Z}$.

Construction of $S^{-1}\mathcal{C}$ for a multiplicative system S in a category $\mathcal{C}$.

Let $S \ni s: X_1 \to X$ and $f: X_1 \to Y$ a morphism in $\mathcal{C}$. Then we call

$X \xleftarrow{s} X_1 \xrightarrow{f} Y$ a left fraction, notation: $s^{-1}f: X \to Y$.

For $S \ni t: Y_1 \to Y$ and $g: X \to Y_1$ a morphism in $\mathcal{C}$ we call

$X \xrightarrow{g} Y_1 \xleftarrow{t} Y$ a right fraction, notation $g t^{-1}: X \to Y$

(The left/right notation here is not uniformly used. Another popular notation is to display fractions as "roofs"

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

or

$$\begin{tikzcd} X \arrow[dr, "g"'] & & Y \arrow[dl, "t"] \\ & Y_1 & \end{tikzcd}$$

)

Following the construction of $\mathbb{Q}$ from $\mathbb{Z}$, the next step is to define equivalence of left (or of right) fractions:

We call two left fractions $s^{-1}f: X \leftarrow X_1 \to Y$ and $t^{-1}g: X \leftarrow X_2 \to Y$ (same X, Y for both) equivalent :⟺ $\exists$ left fraction $r^{-1}h: X \leftarrow X_3 \to Y$ (again same X, Y) and $\exists$ morphisms $\alpha: X_3 \to X_1$ and $\beta: X_3 \to X_2$ such that the diagram is commutative. (Analogously for right fractions.)

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X_3 \arrow[l, "r"'] \arrow[u, "\alpha"] \arrow[r, "h"] \arrow[d, "\beta"] & Y \\ & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

(View this as "expanding fractions":

$\frac{f}{s} = \frac{f\circ\alpha}{s\circ\alpha} = \frac{h}{r} = \frac{g\circ\beta}{t\circ\beta} = \frac{g}{t}$ where "=" is defined by equivalence.)

(Warning: Writing $X \xleftarrow{s} X_1 \xrightarrow{f} Y$ as $\frac{f}{s}$ helps to get an idea what is happening. It does, however, not lead to proofs, since we don't know if $\frac{f}{s}$ is a left fraction or a right fraction.)

We have to check that equivalence is an equivalence relation. Reflexivity and symmetry are obvious, transitivity is not (because of non-commutativity). Since the diagram proving transitivity will be big we write morphisms in S in a different colour, and we display fractions as roofs:

Given

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & & X_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & & & X_3 \arrow[dl, "s_3"'] \arrow[dr, "f_3"] & \\
X & & Y & \sim & X & & Y \arrow[r, phantom, "\sim"] & X & & Y
\end{tikzcd}$$

where the first and the second are equivalent and the second and the third one are equivalent. We have to find an equivalence between the first and the third roof.

We enter the assumption into the picture:

$$\begin{tikzcd}
 & X & & & & Y & \\
X_1 \arrow[ur, "s_1"] \arrow[urrrrr, "f_1"] & & & X_2 \arrow[ull, "s_2"'] \arrow[urr, "f_2"'] & & & X_3 \arrow[ulllll, "s_3"'] \arrow[ul, "f_3"'] \\
 & & U \arrow[ull, "\alpha"] \arrow[uul, "r_1"] \arrow[ur, "\beta"] \arrow[uurrr, "g_1"] & & V \arrow[ul, "\gamma"'] \arrow[urr, "\delta"'] \arrow[uulll, "r_2"] \arrow[uur, "g_2"'] & & \\
 & & & Z \arrow[ul, "h"] \arrow[ur, "t"'] & & &
\end{tikzcd}$$

Here: $r_1 = s_1 \circ \alpha, g_1 = f_1 \circ \alpha,$
$r_1 = s_2 \circ \beta, g_1 = f_2 \circ \beta$
(by: roof 1 ~ roof 2)
and $r_2 = s_2 \circ \gamma, g_2 = f_2 \circ \gamma,$
$r_2 = s_3 \circ \delta, g_2 = f_3 \circ \delta$
(by: roof 2 ~ roof 3)

By the Ore condition, there exists $Z, h: Z \to U$ and $t \in S$: $t: Z \to V$ such that $r_1 \circ h = r_2 \circ t$

(We choose $r_1 \in S$, then we get $t \in S$, but not necessarily $h \in S$)

The equality $r_1 \circ h = r_2 \circ t$ can be rewritten as $s_2 \circ \beta \circ h = s_2 \circ \gamma \circ t$. By the cancellation rule there exists $t' \in S$, $t': Z' \to Z$ such that $\beta \circ h \circ t' = \gamma \circ t \circ t'$. This yields

$$\begin{tikzcd}
 & X & & Y & \\
X_1 \arrow[ur] \arrow[urr] & & & & X_3 \arrow[ulll] \arrow[ul] \\
 & U \arrow[ul] & & V \arrow[uull] \arrow[ur] & \\
 & & Z' \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

or just

$$\begin{tikzcd}
 & X & & Y & \\
X_1 \arrow[ur] \arrow[urrr] & & & & X_3 \arrow[ulll] \arrow[ul] \\
 & & Z' \arrow[ull] \arrow[uul] \arrow[uur] \arrow[urr] & &
\end{tikzcd}$$

which is the diagram in the definition of equivalence

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It remains to check commutativity of the last diagram.
For instance: We need $\underbrace{r_2\circ t}_{\in S}\circ t' : Z' \to X$ to be equal to $s_1\circ\underbrace{(\alpha\circ h\circ t')}_{Z'\to X_1}$, which follows

from $s_1(\alpha\circ h\circ t') = (s_1\circ\alpha)\circ h\circ t' = r_1\circ h\circ t' = r_2\circ t\circ t'$!

Now fractions are well-defined as equivalence classe of roofs. Since fractions will become homomorphisms in the localised category $S^{-1}\mathcal{C}$, we have to define composition, ie multiplication of roofs.

Multiplication of fractions:

Let
$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
and
$$\begin{tikzcd} & Y_1 \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$$
be representatives of fractions.

By the Ore condition,
$$\begin{tikzcd} X_1 \arrow[dr, "f"'] & & Y_1 \arrow[dl, "t"] \\ & Y & \end{tikzcd}$$
can be completed to a commutative square
$$\begin{tikzcd} & W \arrow[dl, "r"'] \arrow[dr, "h"] & \\ X_1 \arrow[dr, "f"'] & & Y_1 \arrow[dl, "t"] \\ & Y & \end{tikzcd}$$

which provides a roof
$$\begin{tikzcd} & & W \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & X_1 \arrow[dl, "s"'] & & Y_1 \arrow[dr, "g"] & \\ X & & & & Z \end{tikzcd}$$
which we define to be the product, and this is associative.

(In terms of fractions, the Ore condition tells us: $f\circ r = t\circ h$ means $\frac{f}{t} = \frac{h}{r}$ and thus $\frac{f}{s}\cdot\frac{g}{t} = \frac{g\circ h}{s\circ r}$.) Writing down diagrams of roofs, one checks that the product is independent of the choice of W, h, r and of the choice of representatives of the given fractions.

Hence, given a category $\mathcal{C}$ with a multiplicative system S, we can define the localisation $S^{-1}\mathcal{C}$ (or: $\mathcal{C}[S^{-1}]$) of $\mathcal{C}$ at S, as a new category, and there is a quotient functor $Q: \mathcal{C}\to S^{-1}\mathcal{C}$ sending X to itself and $f: X\to Y$ to
$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
(ie $f \mapsto \frac{f}{1}$).

$Q: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y) = \{$ roofs: $\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$ $\}$ where $s\in S$, $f\in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

At this point, a serious problem is arising: In a category we require the morphisms $Hom(X,Y)$ to form a set, while the objects may form a class. Since X_1 in

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

can run through the elements of a class (for instance, the isomorphism class of X), $Hom_{S^{-1}\mathcal{C}}(X,Y)$ may not be a set, and in general there is no way to get around this problem. For the derived module categories we are going to define now, there are no set theoretical problems. One can show in several ways that these categories really have sets as $Hom(X,Y)$, and we will see one such argument. For derived categories of abelian categories one may already run into problems.

We still need to check that $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ (constructed as above) has the properties required in Definition 12.5:

(1) Let $s \in S$ (multiplicative system). Then $Q : s \longmapsto$

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}$$

which we can compose with

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "1_X"] & \\ Y & & X \end{tikzcd}$$

to get

$$\begin{tikzcd} & & W \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & & \\ & X \arrow[dl, "1_X"'] & & X \arrow[dr, "1_X"] & \\ X & & & & X \end{tikzcd}$$

(by completing $\begin{tikzcd} X \arrow[dr, "1_X"'] & & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$)

which is the identity on X.

(2) Let F be another functor turning all $s \in S$ into isomorphisms:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C}' \\ S^{-1}\mathcal{C} \arrow[ur, dashed, "\text{want } \hat{F}"'] & \end{tikzcd}$$

We have to define $\hat{F}$ to send X to $F(X)$.

For a fraction

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

we have to define $\hat{F}$ to send it to $F(f) \circ F(s)^{-1}$, and this works.

$\Rightarrow$ Q is the localisation functor.

So far, $\mathcal{C}$ has been a category without additional structure. At this point one has to ask if localisation preserves, or not, additional structure. A positive example is additive structure, which is preserved. We will see later that the localisation we use to define derived categories, does preserve the essential structure in this context (triangulated categories).

12.7 Definition: Let $\mathcal{A}$ be an abelian category, $K(\mathcal{A})$ the homotopy category and Qis the class of quasi-isomorphisms. Then the derived category $D(\mathcal{A})$ is defined to be $K(\mathcal{A})[\mathrm{Qis}^{-1}]$, the localisation of $K(\mathcal{A})$ at Qis.

This definition makes sense (up to set theoretical problems) once we have checked that $S := \mathrm{Qis}$ forms a multiplicative system:
Condition (1) in 12.6 is rather clear: for each X, id_X is a qis, and the product of two qis is a qis.
Condition (2) and Condition (3) are not at all clear. We will check them later, once we know more about the structure of $K(\mathcal{A})$, which $D(\mathcal{A})$ then inherits.

At this point we should list the tasks yet to be performed in the context of defining $D(\mathcal{A})$, at least for $\mathcal{A} = R\text{-Mod}$ (or $A\text{-mod}$, A a finite dimensional algebra):

- Define what a triangulated category is.
- Show that $K(\mathcal{A})$ is triangulated.
- Show that Qis a multiplicative system in $K(\mathcal{A})$.
- Show that localisation preserves the structure of a triangulated category, and hence that $D(\mathcal{A})$ is triangulated as well.
- Get back to the issue of morphisms forming sets.