# Exercises on triangulated categories

(1) Show that the direct sum of two distinguished triangles is a distinguished triangle.

(2) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ be a distinguished triangle.

(a) Show that $f$ is an isomorphism if and only if $Z \simeq 0$.

(b) Assume $h = 0$. Show that $Y \simeq X \oplus Z$.

(3) Let $F : \mathcal{T} \longrightarrow \mathcal{T}'$ be a functor between triangulated categories (i.e. compatible with the triangulated structure). Show that the full subcategory $\mathrm{Ker}(F)$ whose objects satisfy $F(X) \cong 0$ is a triangulated subcategory of $\mathcal{T}$.

(4) Let $\mathcal{T}$ be a category that has all properties of a triangulated category except possibly (TR4). A PP-diagram is a commutative square
$$\begin{tikzcd} Y \arrow[r, "v'"] \arrow[d, "v"'] & Z' \arrow[d] \\ Z \arrow[r] & Y' \end{tikzcd}$$
inducing a distinguished triangle
$$Y \xrightarrow{(-v, v')} Z \oplus Z' \longrightarrow Y' \xrightarrow{\partial}$$

(a) Explain the name "PP-diagram".

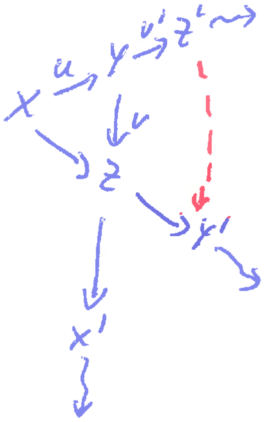
(b) Show that (TR4) is equivalent to the following property

(*) For any PP-diagram with associated triangle as above, there is a commutative diagram
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v'"] \arrow[d, "v"] & Z' \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\ X \arrow[r] & Z \arrow[r] & Y' \arrow[r, "f"] & X[1] \end{tikzcd}$$

where $\partial = -u[1] \circ f$, the rows are distinguished triangles (and the vertical maps define a morphism of triangles).

The proof may use the following steps:

Assume (*) and prove (TR4):

Given  one can try to find  in a PP-diagram
$$\begin{tikzcd} Y \arrow[r, "v'"] \arrow[d, "v"'] & Z' \arrow[d] \\ Z \arrow[r] & Y' \end{tikzcd}$$

by completing $Y \xrightarrow{(-v, v')} Z \oplus Z'$ to a distinguished triangle with cone $W$ and showing $W \cong Y'$!

(*) associates two diagrams with the PP square, one horizontal, the other one vertical. Then commutativities need to be checked.

Assume (TR4) and prove (*):

Given
$$\begin{tikzcd} Y \arrow[r, "v'"] \arrow[d, "v"'] & Z' \arrow[d] \\ Z \arrow[r] & Y' \end{tikzcd}$$
, there is an associated distinguished triangle and one can generate another one,

$X \to Y \xrightarrow{v'} Z' \rightsquigarrow$ by completing $Y \xrightarrow{v'} Z'$.

And (TR4) can be applied to
$$\begin{tikzcd} Y \arrow[r] \arrow[dr, bend right] & Z \oplus Z' \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & Z' \end{tikzcd}$$

How many PP-squares do you see in the octahedron in (TR4)?

And what does the following picture tell you?