Almost split sequences are our most important tool to find many or even all indecomposable representations. They also help us to visualise a module category:

11.17 Definition: The Auslander Reiten quiver of a finite dimensional algebra A is a valued directed graph, i.e. a valued quiver:

- The vertices $[X]$ are the isomorphism classes of indecomposable A-modules X.
- There is an arrow $[X] \to [Y] :\Leftrightarrow \exists$ irreducible map $X \to Y$.
- An arrow $[X] \to [Y]$ gets the valuation $(a,b) :\Leftrightarrow$

$\exists$ left minimal almost split map $X \to Y^b \oplus Z$ (with $Y \nmid Z$) and

$\exists$ right minimal almost split map $X^a \oplus W \to Y$ (with $X \nmid W$).

Notation: $[X] \xrightarrow{(a,b)} [Y]$. For $[X] \xrightarrow{(1,1)} [Y]$ we write $[X] \to [Y]$. We often omit square brackets and just write X.

We often add further information: a dashed line $Z \leftarrow --- X$ means $Z = \tau X = D\,Tr\, X$.

We compute some examples of $A = kQ$.

Example 1: $Q = \bullet$, $A = k$

$X = k$ is projective $\Rightarrow$ the minimal almost split map arriving in X is $\underset{=0}{\text{rad}\, X} \hookrightarrow X$

$X = k$ is injective $\Rightarrow$ the minimal almost split map leaving X is $X \to \underbrace{X/\text{soc}\, X}_{=0}$

For any indecomposable Y, a map $X \to Y$ has to factor through 0, and a map $Y \to X$ also has to factor through 0. In both cases, the map has to be zero, and thus Y has to be zero. $\Rightarrow$ A has AR quiver $\bullet$ where the module is k, the only indecomposable A-module.

Also: $D\,Tr(k) = 0$ since k is projective. $Tr\,D(k) = 0$ since k is injective.

Example 2: $Q = \overset{2}{\bullet} \to \overset{1}{\bullet}$

We know some representations: $S(1) = 0 \to k$, $S(2) = k \to 0$,

$P(1) = S(1)$, $P(2) = k \xrightarrow{1} k$, $I(1) = P(2)$, $I(2) = S(2)$ why?

To find almost split sequences we start almost split maps $\text{rad}\, P \hookrightarrow P$ for P projective:

$0 \hookrightarrow P(1)$ and $P(1) \hookrightarrow P(2)$. $P(1)$ is not injective why not?

$\Rightarrow$ we can compute $Tr\,D\,P(1)$ and look for the almost split sequence starting in $P(1)$:

$P(1) = S(1) \Rightarrow DP(1)$ is a simple right module. Since $P(1)$ is not injective $\Rightarrow$ $DP(1)$ is not projective. $Q^{op} = \bullet \leftarrow \bullet$ and $DP(1) = 0 \leftarrow k$

Compute $Tr(DP(1))$: projective resolution

$$\begin{tikzcd} 0 \leftarrow k \\ k \leftarrow k \arrow[u] \\ k \leftarrow 0 \arrow[u] \end{tikzcd}$$

and apply $Hom_{kQ^{op}}(-, kQ^{op})$ to get

$(0 \leftarrow k)^* \to (k \leftarrow k)^* \to (k \leftarrow 0)^* \to Tr \to 0$

which yields $(0 \to k) \to (k \xrightarrow{1} k) \to Tr\, DP(1) \to 0$

(using that $Hom_B(Be, B) = eB$ and $Hom_B(eB, B) = Be$ for any algebra B, $e = e^2 \in B$)

$\Rightarrow (Tr\, D)P(1) \simeq S(2)$

$\Rightarrow$ there is an almost split sequence $0 \to P(1) \to E \to S(2) \to 0$, where $P(2)$ must be a direct summand of E because $P(1) \hookrightarrow P(2)$ is irreducible.

From $\dim_k E = 2 = \dim_k P(2)$ it follows that the almost split sequence is $0 \to P(1) \to P(2) \to S(2) \to 0$ $(*)$

$P(2)$ is injective $\Rightarrow$ the minimal almost split starting there is

$$P(2) \twoheadrightarrow \big(P(2)/\underbrace{socP(2)}_{=P(1)}\big) \simeq S(2), \text{ which is in } (*)$$

and there is no other almost split map leaving $P(2)$.

$S(2)$ is injective as well and equals its socle $\Rightarrow$ no irreducible map starting there.

What did we get so far? We have found a connected component of the AR quiver

$$\begin{tikzcd} & P(2) \arrow[dr] & \\ P(1) \arrow[ur] & & S(2) \arrow[ll, dashed] \end{tikzcd}$$

(there are no other irreducible maps starting or ending in $P(1)$, $P(2)$ or $S(2)$)

and this component contains all indecomposable modules.

Suppose there exists an indecomposable module X not isomorphic to $P(1)$, $P(2)$ or $S(2)$. There must be a non-zero map $P(1) \to X$ or $P(2) \to X$. By Lemma 11.15, either we reach X by a finite chain of irreducible maps starting at $P(1)$ or $P(2)$ or there is an arbitrarily long chain starting there that can be composed with a map ending in X. But there is no chain of length > 2 $\Rightarrow$ X must be one of $P(1)$, $P(2)$ or $S(2)$.

Result: We have shown that the indecomposable A-modules up to isomorphism are $P(1)$, $P(2)$ and $S(2)$.

This example illustrates a general criterion:

11.18 Corollary: Let $\mathcal{C}$ be a finite connected component of the AR quiver of A and suppose $\mathcal{C}$ contains all indecomposable projective A-modules. Then A has finite representation type and $\mathcal{C}$ contains all indecomposable A-modules.

Proof: Let X be indecomposable and not in $\mathcal{C}$. There must be P projective in $\mathcal{C}$ such that $\mathrm{Hom}_A(P,X) \neq 0$. Since X is not in $\mathcal{C}$ there cannot be a finite chain of irreducible morphisms $P \to \dots \to X$ between indecomposables. Hence, by Lemma 11.15 there are arbitrarily long chains

$$P = X_0 \xrightarrow{f_1} \dots \xrightarrow{f_t} X_t \xrightarrow{g} X$$ between indecomposables, with non-zero composition and $X_0, \dots, X_t$ all in $\mathcal{C}$ and $f_1, \dots, f_t$ all irreducible.

However, $\mathcal{C}$ by assumption is finite. $\Rightarrow$ There is an upper bound for the length of the indecomposables in $\mathcal{C}$. $\Rightarrow$ The Lemma of Harada and Sai (11.14) tells us that for t large enough the composition $f_t \circ \dots \circ f_1$ equals zero.

$\Rightarrow$ X does not exist, $\mathcal{C}$ contains all indecomposables. $\square$

Is it necessary to assume that all indecomposable projectives are in $\mathcal{C}$?

Example 3: $Q = \overset{5}{\bullet} \longrightarrow \overset{4}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{1}{\bullet}$

This is supposed to illustrate that sometimes producing an AR quiver is a kind of needlework. The method we are going to use is called knitting.

First step: Determine the indecomposable projectives or injectives, or their dimension vectors.

$P(1) = 0 \to 0 \to 0 \to 0 \to k \qquad (0,0,0,0,1) = \underline{\dim}\, P(1)$

$P(2) = 0 \to 0 \to 0 \to k \xrightarrow{1} k \qquad (0,0,0,1,1)$

$P(3) = 0 \to 0 \to k \xrightarrow{1} k \xrightarrow{1} k \qquad (0,0,1,1,1)$

$P(4) = 0 \to k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k \qquad (0,1,1,1,1)$

$P(5) = k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k \qquad (1,1,1,1,1)$

$\overset{\shortparallel}{I(1)}$

$I(2) = k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{1} k \to 0 \qquad (1,1,1,1,0)$

$I(3) = k \xrightarrow{1} k \xrightarrow{1} k \to 0 \to 0 \qquad (1,1,1,0,0)$

$I(4) = k \xrightarrow{1} k \to 0 \to 0 \to 0 \qquad (1,1,0,0,0)$

$I(5) = k \to 0 \to 0 \to 0 \to 0 \qquad (1,0,0,0,0)$

$P(1) = \operatorname{rad} P(2)$, $P(2) = \operatorname{rad} P(3)$, $P(3) = \operatorname{rad} P(4)$, $P(4) = \operatorname{rad} P(5)$

$\Rightarrow$ chain of irreducible maps $P(1) \hookrightarrow P(2) \hookrightarrow P(3) \hookrightarrow P(4) \hookrightarrow P(5)$ and all irreducible maps $X \hookrightarrow P(i)$, any i, are in this chain.

Now let $P(j) \xrightarrow{g} X$ be irreducible, X indecomposable, not projective. Then g occurs in the almost split sequence ending in X:

$$0 \to DTrX \to E \to X \to 0 \quad (\text{with } P(j) \mid E)$$

$\Rightarrow \exists$ irreducible map $DTrX \to P(j)$, but there is only one irreducible map ending in $P(j)$: $P(j-1) \hookrightarrow P(j)$ (and $j \geq 2$) $\Rightarrow DTrX \simeq P(j-1)$

Now we start the knitting with $P(1)$. $P(1)$ not injective $\Rightarrow \exists$ almost split sequence $0 \to P(1) \to P(2) \oplus Y \to TrD(P(1)) \to 0$ $(*)$

$\uparrow$ since $\exists$ $P(1) \to P(2)$ irreducible

Y is not projective and even has no projective direct summand, since there is no irreducible map $P(1) \to$ projective $\neq P(2)$. Let Z be any summand of Y. Z not projective $\Rightarrow \exists$ ass $0 \to DTrZ \to E' \to Z \to 0$.

Since $\exists$ irreducible map $P(1) \to Z$, $P(1)$ must be a direct summand of E' $\Rightarrow \exists$ irreducible map $DTrZ \to P(1)$ ↯ (as $P(1)$ is simple projective).

$\Rightarrow Y = 0$ and $(*)$ is $0 \to P(1) \to P(2) \to TrD\,P(1) \to 0$ $(*)$

$\Rightarrow TrD\,P(1) = P(2)/P(1) = S(2)$ simple (and we obtained that without computing $D\,P(1)$ and its transpose).

$\rightsquigarrow$ Now we are one step further in knitting the AR quiver:

$$\begin{tikzcd}
 & & P(5) \\
 & P(2) \arrow[ur, dashed] \arrow[dr] & \\
P(1) \arrow[ur] \arrow[d, equal] & & S(2) \arrow[ll, squiggly, "\tau"] \\
S(1) & &
\end{tikzcd}$$

and we have found another irreducible map starting in $P(2)$. Since $P(2)$ is not injective, there is an ass

$0 \to P(2) \to P(3) \oplus S(2) \oplus Y \to TrD\,P(2) \to 0$ $(**)$

Let $Y' \mid Y$, Y' indecomposable. Y' cannot be projective

$\Rightarrow \exists$ ass $0 \to DTrY \to E'' \to Y \to 0$

As before, $P(2)$ must be a direct summand of E'' $\Rightarrow P(1)$ must be a direct summand of, hence equal to $DTrY$. This implies $Y \simeq S(2)$ and $(**)$ is

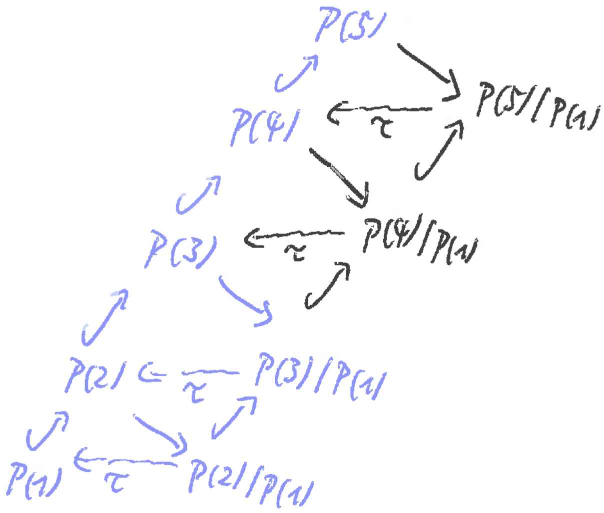
$$(**) \quad 0 \to P(2) \to P(3) \oplus S(2) \to TrD\,P(2) \to 0$$

$\Rightarrow$ Again without computing a transpose we identify $\operatorname{Tr} D\, P(2)$ as

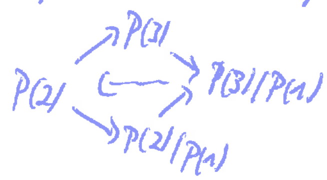
$$\operatorname{Tr} D\, P(2) \simeq P(3) \oplus S(2) / P(2) \simeq 0 \to 0 \to k \xrightarrow{1} k \to 0$$ by a direct computation.

We can avoid the computation by recalling that an irreducible map must be injective or surjective. By dimensions, $S(2) \to \operatorname{Tr} D\, P(2)$ must be injective and $P(3) \to \operatorname{Tr} D\, P(2)$ must be surjective. Counting composition factors, $\operatorname{Tr} D\, P(2)$ must be $P(3)/P(1)$.

Thus we have made another step and have obtained:



Here one may guess the pattern and the black part can be determined exactly in the same way as the mesh

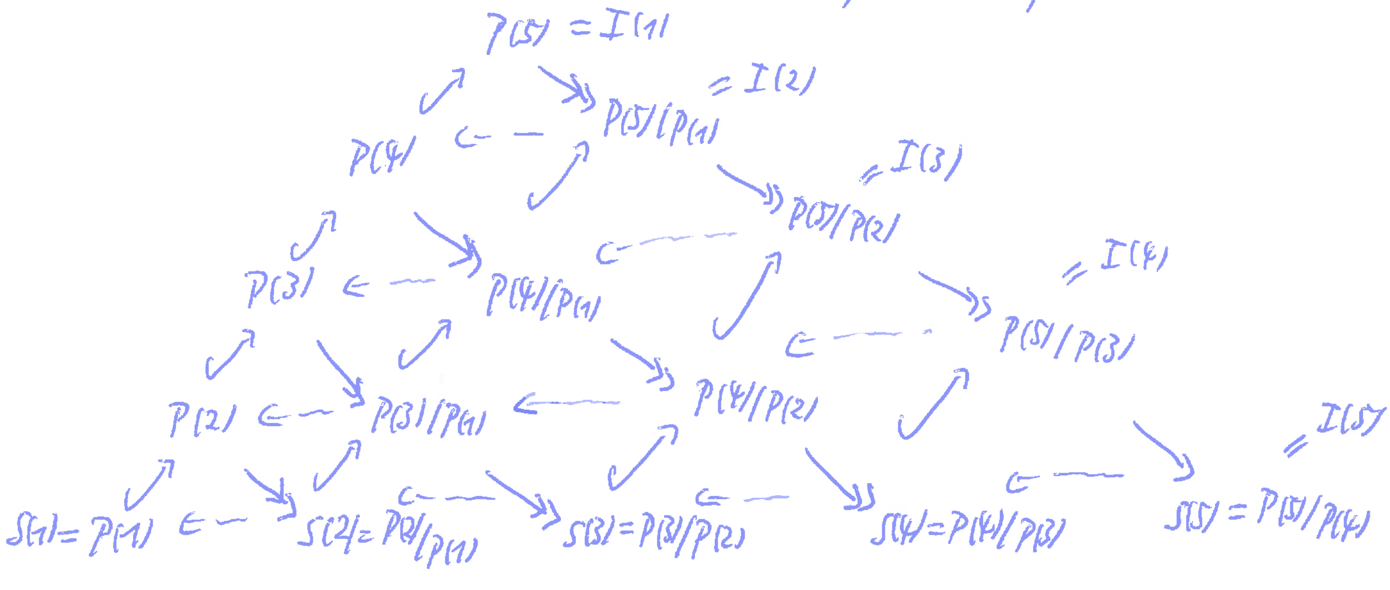


At the upper hand something new happens: $P(5) = I(1)$ is injective and $P(5)/P(1) = I(1)/\operatorname{soc} I(1) = I(2)$ is injective, too. $\Rightarrow$ No ass starts there.

To continue we go to the bottom of the picture and determine the ass starting in $P(2)/P(1) = S(2)$ in the same way as before. It is

$$0 \to S(2) \to P(3)/P(1) \to S(3) \to 0$$

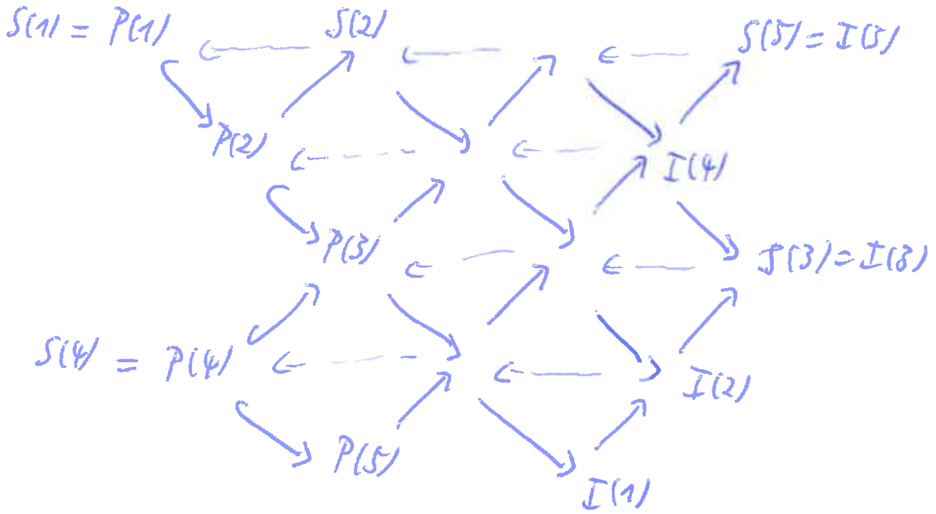
Moving upwards with the same arguments we produce three new meshes, go down again, and so on. Finally we have a complete component:



By Corollary 11.18, all indecomposable modules are in this picture.
To summarise: We started with an algebra of dimension $1+2+3+4+5=15$. We computed the projective modules and then used AR theory to see, without much computation, that up to isomorphism there are exactly 15 indecomposable representations, which we all found. We also found all irreducible morphisms.

In the same way one finds all indecomposable representations of $n \to n-1 \to \dots \to 1$ (their number is $\frac{n(n+1)}{2}$) and the Auslander–Reiten quiver.

This also works with a different orientation on the quiver, for instance
$1 \leftarrow 2 \leftarrow 3 \to 4 \leftarrow 5$ where the AR quiver is



Work out this example in detail.