We are still in the process of proving Theorem 11.4, the main theorem of Auslander – Reiten theory for finite dimensional algebras. Most of the work has been: the Auslander – Reiten formulae 10.4 provide the starting point for proving existence of almost split sequences, the characterisations in Theorem 11.10 simplify verifying that a given sequence is an almost split sequence, and uniqueness follows easily from Lemma 11.11.

Proof of Theorem 11.4:

The proofs of existence in (a) and (b) are dual, we will prove (a).

Let $X \in A\text{-mod}$ be indecomposable and not projective. We have to find an almost split sequence $0 \to D\,Tr\,X \to E \to X \to 0$.

Choosing $Y = X$ in the Auslander – Reiten formula 10.4(a) yields

$$D\,\underline{Hom}_A(X,X) \simeq Ext^1_A(X, D\,Tr\,X)$$

As observed at the end of chapter 10, $\underline{End}_A(X) \neq 0$ as X is not projective, and in particular $id_X \neq 0$ in $\underline{End}_A(X)$.

$\Rightarrow$ in $D\,\underline{End}_A(X)$ there is a non-zero element and hence $Ext^1_A(X, D\,Tr\,X) \neq 0$

$\Rightarrow \exists$ non-split ses $0 \to D\,Tr\,X \to E \to X \to 0$.

There may be many such non-split ses and we have to choose one of them carefully, which we then show to be almost split. To make the choice we use that X is indecomposable. Hence $End_A(X)$ is a local ring. It has a unique maximal ideal that contains exactly the non-invertible endomorphisms of X and that coincides with $rad(End_A(X))$. $S(X) := End_A(X)/rad(End_A(X))$ is, up to isomorphism, the unique simple left $End_A(X)$-module.

Endomorphisms of X that factor through a projective module are in $rad(End_A(X))$

$\Rightarrow$ The epimorphism $End_A(X) \twoheadrightarrow S(X)$ factors through $\underline{End}_A(X)$.

$$\begin{tikzcd} End_A(X) \arrow[rr, two heads] \arrow[dr, two heads] & & S(X) \\ & \underline{End}_A(X) \arrow[ur, two heads] & \end{tikzcd}$$

Dualising yields $DS(X) \hookrightarrow D\,\underline{End}_A(X) \simeq Ext^1_A(X, D\,Tr\,X)$ and $DS(X)$ is the unique simple submodule, i.e. $DS(X) = soc(D\,\underline{End}_A(X))$ (right modules).

We choose any non-zero element in $D\,S(X) \subset \mathrm{Ext}^1_A(X, DTrX)$ and get a non-split ses $\quad (*)\ 0 \to \underbrace{DTrX}_{=\tau X} \xrightarrow{f} E \xrightarrow{g} X \to 0$

We will show that this is an almost split sequence.
The socle of a module contains precisely the elements annihilated by the radical.
In particular, multiplying the sequence $(*)$ by some $\alpha \in \mathrm{rad}(\mathrm{End}(X))$ produces a split exact sequence.

We are going to use 11.10 (b) to check that $(*)$ is almost split.
By 11.5, X indecomposable implies TrX and $DTrX$ indecomposable.
Thus we only have to verify that g is right almost split.

$$\begin{tikzcd} 0 \arrow[r] & DTrX \arrow[r, "f"] & E \arrow[r, "g"] & X \arrow[r] & 0 \\ & & E' := Pb \arrow[u, "\iota"] \arrow[r] & V \arrow[u, "v"] & \end{tikzcd}$$

Let $v: V \to X$ be not split epi. To show: v factors through $g: E \to X$.

Let E' be the pullback.

$\rightsquigarrow$ We get a commutative diagram

$$\begin{tikzcd} (*) & 0 \arrow[r] & DTrX \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r, "g"] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[r] & 0 & \\ (**) & 0 \arrow[r] & DTrX \arrow[r, "f'"'] & E' \arrow[r, "g'"'] \arrow[u, "u"] & V \arrow[r] \arrow[u, "v"'] & 0 & \in \mathrm{Ext}^1_A(V, DTrX) \end{tikzcd}$$

where $(**)$ is obtained from $(*)$ by applying v.

In the special case $V = X$, $v \in \mathrm{rad}(\mathrm{End}(X))$ and $(**)$ splits. We want to show that $(**)$ always splits, whatever V is.

The ses $(*)$ corresponds to $\xi \in D\,\underline{\mathrm{Hom}}_A(X,X)$ and $(**)$ to $\xi' \in D\,\mathrm{Hom}_A(X,V)$

$$\Rightarrow \quad \begin{tikzcd} \xi \in D\,\underline{\mathrm{Hom}}_A(X,X) \arrow[r, "\simeq"] \arrow[d, "v"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Ext}^1_A(X, DTrX) \ni (*) \arrow[d, "v"] \\ \xi' \in D\,\underline{\mathrm{Hom}}_A(X,V) \arrow[r, "\simeq"] & \mathrm{Ext}^1_A(V, DTrX) \ni (**) \end{tikzcd}$$

is a commutative diagram by functoriality of the AR formulae.

What is the meaning of the vertical arrows $\uparrow v$?

On the right hand side, v has been applied to $(*)$ to construct $(**)$.
On the left hand side, applying v is the map dual to

$$\hat{v}: \underline{\mathrm{Hom}}(X,V) \longrightarrow \underline{\mathrm{Hom}}_A(X,X), \quad \alpha \longmapsto v\circ\alpha$$

$\hat{v}$ is not surjective: otherwise $id_X \in \mathrm{Im}(\hat{v})$ and v must be split epi ↯

$\Rightarrow$ The image of $\hat{v}$ is contained in $\mathrm{rad}\, \underline{\mathrm{End}}_A(X)$.
$\Rightarrow$ The map dual to $\hat{v}$ is not injective and its kernel must contain the simple module $DS(X)$. However, ξ was chosen to represent an element in $DS(X)$, hence ξ is mapped to zero $\Rightarrow$ $\xi' = 0$, which means $(**)$ splits.

$\rightsquigarrow$

$$\begin{tikzcd}
(*) & 0 \arrow[r] & D\mathrm{Tr}X \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] & X \arrow[r] & 0 \\
(**) & 0 \arrow[r] & D\mathrm{Tr}X \arrow[r, "f'"] & E' \arrow[r, "g'"] \arrow[u, "u"] & V \arrow[r] \arrow[u, "v"] \arrow[l, bend left, "\exists h"] & 0
\end{tikzcd}$$

splits

$\exists h: V \to E'$ such that $g' \circ h = id_V$

$\Rightarrow v = v\circ id_V = v\circ g'\circ h = g\circ u\circ h \Rightarrow v$ factors through g

$\Rightarrow g$ is right almost split. $\square$

The choice of ξ in the socle fits to the name "almost split" of $(*)$: When we apply any radical map we always get zero, i.e. a split sequence. Any other non-zero element may remain non-zero when applying radical map. So ξ is closer to becoming split than other non-split sequence.

Now a very powerful tool has become available and we can use it for practical and theoretical purposes. We start with a theoretical consequence:

<u>11.12 Definition:</u> Let A be a finite dimensional algebra over a field k.
(a) A has <u>finite representation type</u> $:\Leftrightarrow$ up to isomorphism there are only finitely many indecomposable A-modules.
(b) A has <u>bounded representation type</u> $:\Leftrightarrow \exists N_0 \in \mathbb{N}: \dim M \leq N_0$ for all indecomposable A-modules.

(Here, as always, module means finite dimensional module.)
Finite representation type implies bounded representation type. The first Brauer–Thrall conjecture asserts the converse: bounded type implies finite type. Roiter (1968) and Auslander (1974) proved the conjecture in rather different ways. We will follow Auslander's very transparent proof.

<u>11.13 Theorem</u> (first Brauer–Thrall conjecture): Let A have bounded representation type. Then A has finite representation type.

In other words: When A has infinite representation type (i.e. not finite one), for each $n \in \mathbb{N}$ $\exists\, M(n)$ indecomposable with $\dim M(n) \geq n$.

To get an idea how to prove 11.13 let X, Y be indecomposable and $f: X \to Y$ a homomorphism. If f is not an isomorphism, it may be irreducible. Then it is mono – $\dim X < \dim Y$ – or epi – $\dim X > \dim Y$. If it is neither an isomorphism nor irreducible, then f factors through the minimal almost split maps starting in X or ending in Y. The factors are irreducible or factor further. We will show that there is a bound on the number of factors in such factorisations of non-zero maps between indecomposable modules. So, from X one can only see finitely many Y. But any Y must be visible from at least one indecomposable projective module. Hence, there are only finitely many Y.
Now we are making this precise. One ingredient is independent of AR theory, everything else uses results we already have shown.

Since A has finitely many simple modules, up to isomorphism, we may rephrase the condition to have bounded representation type by: $\exists N_1$ such that each indecomposable A-module has $\underbrace{\text{composition length}}_{=\text{ number of composition factors}} \leq N_1$.

11.14 Theorem (Lemma of Harada and Sai): Let $b \in \mathbb{N}$, $M_1, \dots, M_{2^b}$ be indecomposable A-modules, all with composition length $\le b$ and

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \xrightarrow{f_{2^b-2}} M_{2^b-1} \xrightarrow{f_{2^b-1}} M_{2^b}$$ a chain of non-isomorphisms.

Then $f_{2^b-1} \circ \dots \circ f_2 \circ f_1 = 0$.

Is the assumption indecomposable necessary?

(2^b is the best possible bound)

Proof: We write $\ell(M)$ for the composition length of M.

We are going to prove by induction on $n \le b$:

$$\ell(\mathrm{Im}(f_{2^n-1} \circ \dots \circ f_1)) \le b-n$$

For $n=b$, the result follows.

Induction start $n=1$: f_1 cannot be both injective and surjective

$\Rightarrow \ell(\mathrm{Im}(f_1)) \le b-1$.

From n to $n+1$: We factor the long product into

$$\underbrace{M_1 \xrightarrow{f_1} \dots \xrightarrow{f_{2^n-1}} M_{2^n}}_{g \,=\, f_{2^n-1} \circ \dots \circ f_1} \xrightarrow{f_{2^n}} \underbrace{M_{2^n+1} \xrightarrow{f_{2^n+1}} \dots \xrightarrow{f_{2^{n+1}-1}} M_{2^{n+1}}}_{h \,=\, f_{2^{n+1}-1} \circ \dots \circ f_{2^n+1}}$$

By induction: $\ell(\mathrm{Im}(g)) \le b-n$ and $\ell(\mathrm{Im}\, h) \le b-n$. If one of these lengths is even $\le b-(n+1)$, the claim out follows.

Remaining case: $\ell(\mathrm{Im}(g)) = b-n = \ell(\mathrm{Im}(h))$. Assume $\ell(\mathrm{Im} f) = b-n$, too.

We will show that f_{2^n} must be an isomorphism, which is a contradiction.

This requires some computations:

$$b-n = \ell(\overset{\mathrm{Im}(f)}{\cancel{f}}) = \ell(\mathrm{Im}(h \circ f_{2^n} \circ g)) = \ell(\mathrm{Im}(g) / \mathrm{Im}(g) \cap \mathrm{Ker}(h \circ f_{2^n})) =$$

$$= \ell(\mathrm{Im}(g)) - \ell(\mathrm{Im}(g) \cap \mathrm{Ker}(h \circ f_{2^n})) = b-n - \ell(\mathrm{Im}(g) \cap \mathrm{Ker}(h \circ f_{2^n}))$$

$\Rightarrow \mathrm{Im}(g) \cap \mathrm{Ker}(h \circ f_{2^n}) = 0$

$\mathrm{Im}(h) \supset \mathrm{Im}(h \circ f_{2^n}) \supset \mathrm{Im}(h \circ f_{2^n} \circ g) = \mathrm{Im}(f)$ implies on composition lengths

$b-n \ge \ell(\mathrm{Im}(h \circ f_{2^n})) \ge b-n \Rightarrow \ell(\mathrm{Im}(h \circ f_{2^n})) = b-n = \ell(\mathrm{Im}(g))$

$\Rightarrow \ell(\mathrm{Ker}(h \circ f_{2^n})) = \ell(M_{2^n}) - \ell(\mathrm{Im}(h \circ f_{2^n})) = \ell(M_{2^n}) - \ell(\mathrm{Im}(g))$

$\Rightarrow M_{2^n} = \mathrm{Im}(g) \oplus \mathrm{Ker}(h \circ f_{2^n})$. M_{2^n} is, however, indecomposable $\Rightarrow$ one summand must be zero.

If $Im(g)=0$ then $g=0$ and $f=0$ ✓

We are left with the case $Ker(h\circ f_{2n})=0$, ie $h\circ f_{2n}$ injective $\Rightarrow f_{2n}$ injective.

Want to show: f_{2n} also surjective.

Repeating the above argument for $f_{2n}\circ g$ and h instead of g and $h\circ f_{2n}$ we get $M_{2n+1}=Im(f_{2n}\circ g)\oplus Ker(h)$ and again one summand must be zero.

If $h=0$ we get $f=0$ ✓ Otherwise $M_{2n+1}=Im(f_{2n}\circ g)\Rightarrow f_{2n}\circ g$ surjective $\Rightarrow f_{2n}$ surjective. Altogether f_{2n} isomorphism ↯ □

Now we turn to irreducible maps. There are plenty of these because of the existence of left/right minimal almost split maps.

Let X be indecomposable. Then there exist a left minimal almost split map $f:X\to N$ and a right minimal almost split map $g:M\to X$.

Why? If X is projective, $g:\operatorname{rad}X\to X$ is right minimal almost split. Otherwise we find $g:M\to X$ in the almost split sequence ending at X.

If X is injective, $f:X\to X/\operatorname{soc}X$ is left minimal almost split. Otherwise we find f in the almost split sequence starting at X.

We wish to form long chains of irreducible maps between indecomposable modules and then to make use of 11.14.

11.15 Lemma: Let X and Y be indecomposable and $Hom_A(X,Y)\neq 0$. Suppose there is no chain of s irreducible morphisms between indecomposable modules

$$X\xrightarrow{h_1}\ldots\xrightarrow{h_s}Y \text{ for any } s\leq t.$$

Then there exists a chain of t irreducible morphisms between indecomposable modules and a morphism $g:X_t\to Y$ such that $g\circ f_t\circ\ldots\circ f_1\neq 0$

$$X=X_0\xrightarrow{f_1}\ldots\xrightarrow{f_t}X_t\xrightarrow{g}Y$$

and there also exists a chain of t irreducible morphisms between indecomposable modules and a morphism $f:X\to Y_0$ such that $g_t\circ\ldots\circ g_1\circ f\neq 0$

$$X\xrightarrow{f}Y_0\xrightarrow{g_1}\ldots\xrightarrow{g_t}Y_t=Y$$

Proof: We prove the first assertion by induction on t. For $t=0$ there is nothing to show.

Inductionstep: There exists a chain $X=X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_t} X_t \xrightarrow{g} Y$ with all f_i irreducible, all X_i indecomposable and $g\circ f_t \circ \cdots \circ f_1 \neq 0$.

If g is an isomorphism, then $g\circ f_t$ is irreducible and $X=X_0 \xrightarrow{f_1} \cdots \to X_{t-1} \xrightarrow{g\circ f_t} Y$ is a chain of irreducible maps – which by assumption doesn't exist.

X_t and Y are indecomposable $\Rightarrow$ g is not split epi or split mono, hence not an isomorphism.

Consider a left minimal almost split map $X_t \xrightarrow{h=(h_1,\dots,h_n)} Z_1\oplus\cdots\oplus Z_n$ (all h_i irreducible, all Z_i indecomposable) $\Rightarrow$ g factors:

$$\begin{tikzcd} X_t \arrow[rr, "{h=(h_1,\dots,h_n)}"] \arrow[dr, "g"'] & \arrow[d, phantom, "\circlearrowleft"] & Z_1\oplus\cdots\oplus Z_n \arrow[dl, "{\exists j=(j_1,\dots,j_n)}"] \\ & Y & \end{tikzcd}$$

$g=j\circ h$, and $g\circ f_t\circ\cdots\circ f_1\neq 0$ implies

$\exists i: j_i\circ h_i\circ f_t\circ\cdots\circ f_1\neq 0 \Rightarrow X=X_0\xrightarrow{f_1}\cdots\xrightarrow{f_t} X_t\xrightarrow{h_i} Z_i\xrightarrow{j_i} Y$ is the desired chain of $t+1$ irreducible maps with non-zero compositions. □

Now we are ready to give Auslander's proof of the first Brauer–Thrall conjecture.

<u>Proof of Theorem 11.13.</u>

Let A be of bounded representation type and let b be an upper bound for $\ell(M)$, M indecomposable. Want to show: A has finite representation type.

Let X, Y be indecomposable A-modules and $\operatorname{Hom}_A(X,Y)\neq 0$.

<u>Claim:</u> $\exists$ chain $X=X_0\xrightarrow{f_1}\cdots\xrightarrow{f_t} X_t=Y$ of irreducible maps with $t<2^b-1$.

Proof of claim: Assume there is no such chain. Then Lemma 11.15 implies the existence of irreducible maps $f_1,\dots,f_{2^b-2}$ and g such that

$$X=X_0\xrightarrow{f_1}\cdots\xrightarrow{f_{2^b-2}} X_{2^b-1}\xrightarrow{g} Y \text{ with non-zero composition.}$$

None of the f_i is an isomorphism and g isn't an isomorphism either. All X_i and Y are indecomposable $\Rightarrow$ By the Harada–Sai Lemma, $g\circ f_{2^b-2}\circ\cdots\circ f_1=0$ ↯

So the claim is true.

Hence, Y can be reached from X by a chain of not more than 2^b-1 irreducible maps. Irreducible maps are summands of minimal almost split maps.

$\Rightarrow$ From a given X one can only "see" finitely many indecomposable Y (up to isomorphism) by $\operatorname{Hom}_A(X,Y)\neq 0$.

The regular module $_AA$ is a finite direct sum of indecomposable projective modules. $\Rightarrow$ $\mathrm{Hom}_A(A,Y) \neq 0$ for only finitely many Y. But $\mathrm{Hom}_A(A,Y) \cong Y$
$\Rightarrow$ A has finite representation type. $\square$

This was a very pretty proof that demonstrates how natural the concept of irreducible map is. And this proof shows more than just finite representation type:

11.16 Corollary: Let A be of finite representation type and $f: X \to Y$, X, Y indecomposable, a non-zero map. Then f is a linear combination of finite products of isomorphisms and irreducible maps.
Proof?

If one is able to determine the irreducible maps $P \to X$ starting in projective modules, the irreducible maps starting in the modules X_i and so on, then after finitely many steps all chains starting in some P compose to 0 and one has found all A-modules.

There is a second Brauer-Thrall conjecture: Let A be finite dimensional and the ground field k be algebraically closed. Then either A is of finite representation type or there exist infinitely many $d \in \mathbb{N}$ such that there exist infinitely many pairwise non-isomorphic indecomposable A-modules M with length $\ell(M) = d$.
This was shown by Bautista (On algebras of strongly unbounded representation type, 1985) for char $k \neq 2$. In 1986, Bongartz extended the result to general $k = \bar{k}$.
Bautista's proof heavily relies on results of Bautista, Gabriel, Roiter and Salmeron (Representation-finite algebras and multiplicative bases, 1985).
The BT II conjecture is open for general k. For finite k it doesn't make sense:
Then M with $\dim M = d$ is a finite set and there can be only finitely many A-module structures on a finite set.
Smalø showed in 1980 that finding one d with infinitely many M is enough.

Much later, Bongartz (Indecomposables live in all smaller lengths, 2013) proved:

- Let A be finite dimensional over $k=\bar{k}$. If M is indecomposable and $\ell(M)=n\geq 2$, then there exists an indecomposable N with $\ell(N)=n-1$.
- Let A be finite dimensional over $k=\bar{k}$. Then A is of finite representation type $\Leftrightarrow$ $\exists n\in\mathbb{N}$ such that there is no indecomposable M with $\ell(M)=n$. (One may choose $n=2\cdot\dim A+1000$.)

(The second statement is a consequence of the first one. The first one uses the results by Bautista, Gabriel, Roiter and Salmeron that were also used in the proof of BT II.)