Preview

In Chapter 10 we have established two rather deep (and a bit mysterious) formulae, Auslander's defect formula and the Auslander–Reiten formulae. The latter have told us how to find, for any non-projective indecomposable module X, another module $D\,Tr\,X$ such that there exists a non-split ses

$$(*) \quad 0 \to D\,Tr\,X \to E \to X \to 0, \text{ for some } E$$

This is the basis of Auslander–Reiten theory, which we are developing in this chapter.

A short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is split if and only if the identity on Z factors through g:

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ & Z \arrow[ul, dashed, "\exists g'"] \arrow[u, "id_Z"'] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

($g \circ g' = id_Z$ implies g split epi)

if and only if the identity on X factors through f: $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[dl, dashed, "\exists f'"] \\ X & \end{tikzcd}$$

($id_X = f' \circ f$ implies f split mono)

Many other maps $V \to Z$ may be factor through g without forcing the ses to split, and similarly many other maps $X \to U$ may factor through f without forcing the ses to split.

Questions: Can there be a morphism $0 \to X \xrightarrow{f} Y$ that is not split mono such that all other non-split monos leaving X factor through f? (And dually for g not split epi.) This would generate a ses $0 \to X \xrightarrow{f} Y \to \mathrm{Coker}(f) \to 0$ that is not split exact, but "almost" such that plenty of maps factor through f except those (like id_X) that would force the sequence to split.

If f exists then plenty of maps leaving X have the form $f' \circ f$, since they factor, and this factorisation is likely to be non-trivial. If f itself factors non-trivially, say $f = f_2 \circ f_1$, then it may be hard to factor f_1 through f. So, f should be irreducible (have no non-trivial factorisations).

Question: Do "irreducible" morphisms exist?

Hoping for answers to these questions may feel like dreaming of a miracle to happen. However, this miracle has already started to happen in Chapter 10. We will abstractly define a class of "almost split" sequences by requiring the maps to be like f (or g) above, that is, almost all other maps factor through them. In 11.10 we will see that these maps also are "irreducible" and all "irreducible" maps occur as their summands. And then we check that the ses like (*) above are exactly the "almost split" sequences, which therefore exist.

The main theorem, 11.4, about existence and uniqueness of "almost split" sequences is a very strong tool, which we can immediately use to address the questions raised in the preview to chapter 10.
Moreover, we are suddenly to compute all indecomposable finite dimensional modules (up to isomorphism) of many finite dimensional algebras. Moreover, we find a nice and practical visualisation of A-mod.

Auslander–Reiten theory is a major application of homological and categorical tools to representation theory.