§ 11. Almost split sequences

The zero element in $\mathrm{Ext}_A^1(Z,X)$ is represented by any split exact sequence

$$(*) \quad 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

By definition of split exact, there exists a module homomorphism

$$h: Z \to Y \text{ such that } g\circ h = id_Z$$

(Then we say that g is a split epimorphism).

By Lemma 1.3, equivalently there exists $j: Y \to X$ such that $j\circ f = id_X$

(and we call f a split monomorphism).

Moreover $Y \simeq X \oplus Z$ and more precisely, $Y = f(X) \oplus Y'$ for some $Y' \leq Y$, and f can be written as $X \xrightarrow{\binom{f}{0}} f(X) \oplus Y'$! why?

Then every homomorphism $\alpha: X \to M$ (M any A-module) factors through Y:

(more precisely, through f)

$$\begin{tikzcd} X \arrow[r, "\binom{f}{0}"] \arrow[d, "\alpha"'] & Y = f(X) \oplus Y' \arrow[dl, "(\alpha\circ f^{-1}\ \ 0)"] \\ M & \end{tikzcd}$$

Conversely, if $f: X \to Y$ is any morphism and it has the property that each $\alpha: X \to M$ (any M) factors through f, then f is split mono:

Choose $M = X$ and $\alpha = id_X$ $\Rightarrow$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha = 1_X"'] & Y \arrow[dl, dashed, "\exists j"] \\ M = X & \end{tikzcd}$$

and $j\circ f = id_X$, which implies f is split mono.

So we have shown: $f: X \to Y$ is split mono $\Leftrightarrow$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\forall \alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[dl, dashed, "\beta"] \\ \forall M & \end{tikzcd}$$

Analogously one shows:

$g: Y \to Z$ is split epi $\Leftrightarrow$

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, phantom, "\circlearrowright"] & Z \\ \forall N \arrow[u, dashed, "\exists \beta"] \arrow[ur, "\forall \alpha"'] & \end{tikzcd}$$

Now we are going to modify the definitions of split mono, split epi and split exact sequence (without knowing if there are any examples).

11.1 Definition: Let A be an algebra and X, Y, Z A-modules.

(a) A morphism $f: X \to Y$ is left minimal $:\Leftrightarrow$ $\forall \alpha \in \mathrm{End}(Y)$:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, "\alpha"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Y \end{tikzcd} \Rightarrow \alpha \in \mathrm{Aut}(Y)$$

(b) A morphism $g: Y \to Z$ is right minimal $:\Leftrightarrow$ $\forall \alpha \in \mathrm{End}(Y)$:
$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ Y \arrow[u, "\alpha"] \arrow[ur, "g"'] \arrow[ur, phantom, "\circlearrowleft", bend left] & \end{tikzcd} \Rightarrow \alpha \in \mathrm{Aut}(Y)$$

(c) A morphism $f: X \to Y$ is left almost split $:\Leftrightarrow$

f is not split mono and $\forall u: X \to U$, u not split mono $\exists \tilde{u}$ such that
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & \circlearrowleft & Y \\ & U \arrow[ur, dashed, "\exists \tilde{u}"'] & \end{tikzcd}$$
u factors through f: $u = \tilde{u} \circ f$

(d) A morphism $g: Y \to Z$ is right almost split $:\Leftrightarrow$

g is not split epi and $\forall v: V \to Z$, v not split epi $\exists \tilde{v}$ such that
$$\begin{tikzcd} Y \arrow[rr, "g"] & \circlearrowleft & Z \\ & V \arrow[ul, dashed, "\tilde{v}"] \arrow[ur, "v"'] & \end{tikzcd}$$
v factors through g: $v = g \circ \tilde{v}$

(e) A morphism $f: X \to Y$ is left minimal almost split $:\Leftrightarrow$ f satisfies (a) and (c)

(f) A morphism $g: Y \to Z$ is right minimal almost split $:\Leftrightarrow$ g satisfies (b) and (d).

In (c) it is necessary to exclude u split mono, otherwise we can choose for instance $U = X$ and $u = 1_X$, and get f split mono as above.

Example: Let $P = Ae$ indecomposable projective for A finite dimensional. Then the inclusion $g: \mathrm{rad}(Ae) \hookrightarrow Ae$ is right minimal almost split:

g is not surjective $\Rightarrow$ g is not epi, hence not split epi

Consider
$$\begin{tikzcd} \mathrm{rad}(Ae) \arrow[r, hook, "g"] & Ae \\ V \arrow[ur, "v"'] & \end{tikzcd}$$
for v not split epi. Then v is not surjective (otherwise it splits as Ae is projective)

$\Rightarrow$ $\mathrm{Im}(v) \subset \mathrm{rad}(Ae)$, since $\mathrm{rad}(Ae)$ is the unique maximal submodule of Ae $\Rightarrow$ v factors through g and $\exists \tilde{v}$:
$$\begin{tikzcd} \mathrm{rad}(Ae) \arrow[r, hook, "g"] & Ae \\ V \arrow[u, dashed, "\exists \tilde{v}"] \arrow[ur, "v"'] & \end{tikzcd}$$

define $\tilde{v}$

Finally we check that g is right minimal:

Consider
$$\begin{tikzcd} \mathrm{rad}(Ae) \arrow[r, hook, "g"] & Ae \\ \mathrm{rad}(Ae) \arrow[u, "\alpha"] \arrow[ur, "g"'] \arrow[ur, phantom, shift left=4, "\circlearrowleft"] & \end{tikzcd}$$
with $\underset{\overset{\shortparallel}{g\circ 1_{\mathrm{rad}(Ae)}}}{g} = g\circ\alpha$. g injective $\Rightarrow \alpha = 1_{\mathrm{rad}(Ae)}$ $\Rightarrow \alpha$ is right minimal

There is a "dual" example $I \to I/\mathrm{soc}(I)$ for I indecomposable injective.

11.2 Definition:

A short exact sequence
$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \quad (*)$$
is called <u>almost split sequence</u> (or <u>Auslander–Reiten sequence</u>) :⇔ f is left minimal almost split and g right minimal almost split.

Unfortunately, we do not yet know a single example of such a sequence. f is not allowed to be split mono ⇒ $(*)$ cannot be split exact. The only right minimal almost split g we know ($g: \mathrm{rad}(Ae) \hookrightarrow Ae$) is not surjective, hence does not fit into a sequence $(*)$.

There are some clear restrictions on the existence of almost split sequences: X cannot be injective and Z cannot be projective.

Moreover:

11.3 Lemma:

(a) Let $f: X \to Y$ be left almost split. Then X is indecomposable.

(b) Let $g: Y \to Z$ be right almost split. Then Z is indecomposable.

Proof: The two assertions are dual to each other, we only prove (a):

Assume X decomposes non-trivially into $X = X_1 \oplus X_2$, both $\neq 0$. Then the projections $p_1: X \twoheadrightarrow X_1$ and $p_2: X \twoheadrightarrow X_2$ are not mono, hence not split mono.

Consider
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[dl, dashed, "\exists \tilde{u}_1"] \\ X_1 & \end{tikzcd}$$
and
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_2"'] & Y \arrow[dl, dashed, "\exists \tilde{u}_2"] \\ X_2 & \end{tikzcd}$$

$\rightsquigarrow X \xrightarrow[\underset{\binom{p_1}{p_2}}{\shortparallel}]{\mathrm{Id}_X} X = X_1 \oplus X_2$ factors as $\begin{pmatrix}\tilde{u}_1\circ f\\ \tilde{u}_2\circ f\end{pmatrix} = \begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\end{pmatrix}\circ f \Rightarrow f$ is split mono ↯ □

In 1975, Auslander and Reiten published the definition of almost split sequences and proved the following main theorem (in the article "Representation theory of Artin algebras III. Almost split sequences."):

11.4 Theorem: Let A be a finite dimensional algebra.

(a) Let $X \in A$-mod be indecomposable and not projective. Then there exists an almost split sequence $0 \to D\,Tr\,X \to E \to X \to 0$.
If $0 \to Z \to Y \to X \to 0$ is an almost split sequence also ending in X, then there is an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & D\,Tr\,X \arrow[r] \arrow[d, phantom, "\wr"] & E \arrow[r] \arrow[d, phantom, "\wr"] & X \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\
0 \arrow[r] & Z \arrow[r] & Y \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

(b) Let $Y \in A$-mod be indecomposable and not injective. Then there exists an almost split sequence $0 \to Y \to F \to Tr\,D\,Y \to 0$.
If $0 \to Y \to Z \to X \to 0$ is an almost split sequence also starting in Y, then there is an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, phantom, "\wr"] & F \arrow[r] \arrow[d, phantom, "\wr"] & Tr\,D\,Y \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

(So, almost split sequences exist, whenever one can hope they exist, and they are unique.
X not projective means $Ext^1_A(X,-) \neq 0$. 11.4(a) gives an explicit module with $Ext^1_A(X, D\,Tr\,X) \neq 0$.
Y not injective means $Ext^1_A(-,Y) \neq 0$. 11.4(b) gives an explicit module with $Ext^1_A(Tr\,D\,Y, Y) \neq 0$.)

$D\,Tr$ is called Auslander-Reiten translation. Notation: $\tau := D\,Tr$.
$Tr\,D$ is called inverse Auslander-Reiten translation. Notation: $\tau^- = \tau^{-1} = Tr\,D$.

Warning: τ and τ^- are, in general, not functors and they are not always inverse to each other on modules. For X projective, $DTr(X)=0$, and for Y injective, $TrD(Y)=0$, check

The proof of 11.4 will take some time.

Plan of the proof of 11.4:

(a) There is a lack of symmetry in the assertions: X is required (in (a)) to be indecomposable, $DTr\,X$ is not. And in (b), Y is required to be indecomposable, but $Tr\,DY$ is not. But Lemma 11.3 tells us, $DTr\,X$ and $TrD\,Y$ must be indecomposable, when they occur in almost split sequences.
We will show that $DTr\,X$ is indecomposable for X indecomposable and not projective, and $TrD\,Y$ is indecomposable for Y indecomposable and not injective.

(b) By step (a), the end terms of the candidate almost split sequences are indecomposables and there is symmetry in the assertions. In definition 11.2, the map f is required to be left minimal almost split and g is right minimal almost split. These two conditions are symmetric, and we will show that it is enough to check one of them (assuming that both end terms are indecomposable). More generally, we will characterise almost split sequences in several equivalent ways, which shows that only part of the conditions in 11.2 has to be checked. Another equivalent characterisation will provide a rather different and much easier to visualise description of the maps in an almost split sequence.

(c) The characterisations in step (b) are not needed for uniqueness of almost split sequences, which just is easy to show directly.

(d) Steps (a) and (c) are short. Step (b) is lengthy, but not difficult. The hard part in proving 11.4 is existence of almost split sequences. Fortunately, we have already seen the main idea in chapter 10. The Auslander-Reiten formulae allow to show existence of non-split short exact sequences and we will see that the example at the end of chapter 10 produces the almost split sequences we want. We just have to apply the characterisations in step (b).

Recall: For $M \in A$-mod with projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$, applying $\mathrm{Hom}_A(-, A)$ yields

$$0 \to \mathrm{Hom}_A(M, A) \xrightarrow{p_0^t} \mathrm{Hom}_A(P_0, A) = P_0^t \xrightarrow{p_1^t} \mathrm{Hom}_A(P_1, A) = P_1^t \longrightarrow \mathrm{Tr}\, M = \mathrm{Coker}(p_1^t) \to 0$$

11.5 Proposition: Let X and Y be indecomposable A-modules. Then:

(a) $\mathrm{Tr}\, M$ has no non-zero projective direct summand.

(b) If X is not projective and $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X \to 0$ a minimal projective presentation, then $P_0^t \to P_1^t \to \mathrm{Tr}\, X \to 0$ is a minimal projective presentation of $\mathrm{Tr}\, X$.

(c) X is projective $\Leftrightarrow \mathrm{Tr}\, X = 0$

(d) If X is not projective, then $\mathrm{Tr}\, X$ is indecomposable and $\mathrm{Tr}(\mathrm{Tr}\, X) \simeq X$.

(e) If X and Y are not projective, then $X \simeq Y \Leftrightarrow \mathrm{Tr}\, X \simeq \mathrm{Tr}\, Y$.

Proof: (c) Let X be projective. Then $0 \to X \to X \to 0$ is a minimal projective presentation $\Rightarrow P_1^t = 0 \Rightarrow \mathrm{Tr}\, X = 0$. Conversely, suppose $\mathrm{Tr}\, X = \mathrm{Coker}(p_1^t) = 0$. Then p_1^t is surjective: $P_0^t \xrightarrow{p_1^t} P_1^t \to 0$. P_0^t and P_1^t are projective (right modules, when X is a left module) $\Rightarrow p_1^t$ is split epi $\Rightarrow p_1$ is split mono, that is, $P_1 | P_0$. But the presentation is minimal $\Rightarrow P_1 = 0$ and $P_0 = M$.

(b) $P_0^t \to P_1^t \to \mathrm{Tr}\, X \to 0$ is a projective presentation. Assume it is not minimal. Then P_0^t or P_1^t have direct summands that are not needed.

First case: $P_1^t = Q_1 \oplus Q_2$, $Q_1 \to \mathrm{Tr}\, X \to 0$ the projective cover and Q_2 unnecessary. By Proposition 6.7, we may assume that Q_2 maps to zero and the kernel of $P_1^t \to \mathrm{Tr}\, X$ has Q_2 as a direct summand. Then Q_2 is a direct summand of the image of p_1^t, hence of P_0^t, and $Q_2 \xrightarrow{id} Q_2$ is a direct summand of $P_0^t \to P_1^t$.
$\Rightarrow Q_1^t \xrightarrow{id} Q_1^t$ is a direct summand of $P_1 \to P_0$, which is a minimal presentation ↯

Second case: $P_0^t = Q_3 \oplus Q_4$ with unnecessary summand $Q_4 \subset \mathrm{Ker}(p_1^t) \Rightarrow$ $Q_4 \to 0$ is a direct summand of $P_0^t \to P_1^t \Rightarrow 0 \to Q_4^t$ is a direct summand of $P_1 \xrightarrow{p_1} P_0$, hence Q_4^t is a direct summand of X ↯ (check details)

(d) Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X \to 0$ be a minimal projective presentation. Then by (b), $P_0^t \to P_1^t \to \mathrm{Tr}\, X \to 0$ is a minimal projective presentation. X not projective $\Rightarrow$ $P_1 \neq 0$ and $P_0 \neq 0$. Applying transposing to $P_0^t \to P_1^t \to \mathrm{Tr}\, X \to 0$ yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Tr}X)^t \arrow[r] & (P_1^t)^t \arrow[r, "(p_1^t)^t"] \arrow[d, "\wr"] & (P_0^t)^t \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Coker}((p_1^t)^t) = \mathrm{Tr}(\mathrm{Tr}X) \arrow[r] \arrow[d] & 0 \\
 & & P_1 \arrow[r, "p_1"] & P_0 \arrow[r] & \mathrm{Coker}(p_1) = X \arrow[r] & 0
\end{tikzcd}$$

The rows are exact and the two vertical isomorphisms imply the map between cokernels to be an isomorphism, too. So, $X \simeq \mathrm{Tr}(\mathrm{Tr}X)$.

(a) and (e) are consequences of (d). □

This was step (a) in the proof of 11.4.

The maps in almost split sequences are left or right almost split, which means that other maps have to factor through these particular maps: When

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \\
 & & M \arrow[u, "\exists \tilde{g}"] \arrow[ur, "h"'] \arrow[u, phantom, shift left=4, "\circlearrowleft"] & &
\end{tikzcd}$$

(non-split)

is an almost split sequence and $h: M \to Z$ is not split epi, then h must factor through g (and similarly for $h': X \to M$).

So, if $h: M \to N$ is a homomorphism, M not injective and N not projective, then h must factor through $f: M \to E$ in the almost split sequence starting in M, and through $g: E' \to N$ in the ass ending in N.

Such a factorisation can be trivial: N can be a direct summand of E and $E \xrightarrow{\pi} N$ the projection onto this summand.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "h"'] & E \arrow[d, "\pi"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & N
\end{tikzcd}$$

But E can have only finitely many summands (because of the theorem of Krull–Remak–Schmidt)

⇒ h often will factor through f or g in a non-trivial way.

What do we learn from that? Well, when we find $h: M \to N$ ← indecomposable not projective

↑ indec not injective

and h is not an isomorphism, but also does not admit a non-trivial factorisation through a left or right almost split map, then it must be a summand of such a map!

⟶ Idea: study such maps without non-trivial factorisations and relate them with the left or right almost split maps, which we will understand better in this way.

not necessarily indecomposable

11.6 Definition: Let $f: X \to Y$ be a morphism A-mod. Then f is called **irreducible** (or irreducible map) :⟺ f is neither split mono nor split epi and for each factorisation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & \circlearrowleft & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

(for any Z) g is split mono or h is split epi.

In this definition, Z can be chosen freely. For instance, choose $Z = \mathrm{Im}(f)$ and factor f as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & \circlearrowleft & Y \\ & \mathrm{Im}(f) \arrow[ur, "\text{inclusion}"'] & \end{tikzcd}$$

If f is irreducible, the induced map $\bar{f}: X \to \mathrm{Im}(f)$ (with $\bar{f}(x_0) = f(x_0)$ for $x_0 \in X$) must be split mono or the inclusion: $\mathrm{Im}(f) \hookrightarrow Y$ must be split epi. Since $\bar{f}$ by definition is surjective and the inclusion is injective, this means: $\bar{f}$ or inclusion must be an isomorphism.

⇒ f must be injective or surjective (but of course not both at the same time).

So: an irreducible map always is injective or surjective.

Example of an irreducible map: $P = Ae$ indecomposable projective, $\mathrm{rad}(Ae) \neq 0$, $f: \mathrm{rad}(Ae) \hookrightarrow Ae$ the inclusion. Let $S := \mathrm{Coker}(f) = Ae/\mathrm{rad}(Ae)$, simple.

Claim: f is irreducible.

Check the conditions: f not surjective ⇒ f not split epi

P indecomposable ⇒ f not split mono

Suppose there is a factorisation

$$\begin{tikzcd} X = \mathrm{rad}(Ae) \arrow[rr, hook, "f"] \arrow[dr, "g"'] & & Y = Ae \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

If h is surjective it splits as Y is projective.

Otherwise $\mathrm{Im}(h) \subsetneqq Y$.

⇒ $\mathrm{Im}(h) \subset X = \mathrm{rad}(Ae)$, which is the unique maximal submodule.

⇒ h factors through f:

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "\exists j"'] & \circlearrowleft & Y = Ae \\ & \mathrm{rad}(Ae) \arrow[ur, "f"'] & \end{tikzcd}$$

$h = f \circ j \Rightarrow f = h \circ g = f \circ j \circ g$

$\Rightarrow j \circ g = \mathrm{id}_X$ since f is injective

⇒ g is a split monomorphism

Recall that f also is an example of a right minimal split map.

Since we want to compare irreducible maps with maps in almost split sequences, we should consider short exact sequences in which irreducible maps do occur. Such sequences with at least one irreducible map do exist, as f irreducible implies f injective or surjective:

$f: X \to Y$ irreducible, f injective $\Rightarrow \exists$ ses $0 \to X \xrightarrow{f} Y \to \mathrm{Coker}(f) \to 0$

$g: Y \to Z$ irreducible, g surjective $\Rightarrow \exists$ ses $0 \to \mathrm{Ker}(g) \to Y \xrightarrow{g} Z \to 0$

In these sequences, one map is irreducible. What can we say about the other map?

11.7 Proposition: Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence of A-modules, which does not split.

(a) f is irreducible $\Leftrightarrow$ $\forall \varphi: M \to Z$ $\exists \alpha: M \to Y$ such that $\varphi = g \circ \alpha$
or $\exists \beta: Y \to M$ such that $g = \varphi \circ \beta$

(in diagrams:
$$\begin{tikzcd} Y \arrow[r, "g"] & Z \arrow[r] & 0 \\ & M \arrow[ul, "\exists\alpha"] \arrow[u, "\forall\varphi"'] \arrow[ul, phantom, bend right, "\circlearrowright"] \end{tikzcd}$$
or
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "\exists\beta"'] & Z \arrow[r] & 0 \\ & M \arrow[u, "\forall\varphi"'] \arrow[u, phantom, shift left=3, "\circlearrowleft"] \end{tikzcd}$$
)

(b) g is irreducible $\Leftrightarrow$ $\forall \varphi: X \to N$ $\exists \alpha: Y \to N$ such that $\varphi = \alpha \circ f$
or $\exists \beta: N \to Y$ such that $f = \beta \circ \varphi$

(in diagrams:
$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\forall\varphi"'] & Y \arrow[dl, "\exists\alpha"] \\ & N \arrow[ur, phantom, shift right=3, "\circlearrowright"] \end{tikzcd}$$
or
$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\forall\varphi"'] & Y \\ & N \arrow[ur, dashed, "\exists\beta"'] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] \end{tikzcd}$$
)

So, each of the maps f or g "knows" if the other one is irreducible. To see, that both cases can occur consider two extreme cases in (a):

Given $\varphi = id_Z : Z \to Z$, existence of α implies g split epi $\Rightarrow$ f split mono ↯, hence only β can exist.

Given $\varphi = 0 : M \to Z$ (any M), existence of β implies $g = 0 \Rightarrow f$ isomorphism ↯, hence only α can exist.

So one cannot avoid allowing both cases.

Proof of 11.7: Using k-duality $\mathrm{Hom}_k(-,k)$ the two assertions are seen to correspond. Therefore, we only prove (a).

Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be given.

"⇒": Let f be irreducible. Let $\varphi: M \to Z$ be any map.

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ & M \arrow[u, "\varphi"'] \end{tikzcd}$$

This diagram invites us to form a pullback, yielding a commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\sigma"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r, "\tau"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\ulcorner\ \circlearrowleft"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

(E = pullback)

$\Rightarrow f = \xi \circ \sigma$

$f = \xi \circ \sigma$ is irreducible $\Rightarrow$ σ split mono or ξ split epi

First case: σ split mono $\Rightarrow$ $0 \to X \to E \to M \to 0$ splits $\Rightarrow$ τ is split epi, that is $\exists\, \mu: M \to E$ such that $\tau \circ \mu = \mathrm{id}_M$.

$\Rightarrow \varphi = \varphi \circ \mathrm{id}_M = (\varphi \circ \tau) \circ \mu = (g \circ \xi) \circ \mu = g \circ (\xi \circ \mu) = g \circ \alpha$ for $\alpha := \xi \circ \mu : M \to Y$, so α exists as desired

Second case: ξ split epi $\Rightarrow \exists\, \delta: Y \to E$ such that $\mathrm{id}_Y = \xi \circ \delta$

$\Rightarrow g = g \circ \mathrm{id}_Y = g \circ (\xi \circ \delta) = (g \circ \xi) \circ \delta = (\varphi \circ \tau) \circ \delta = \varphi \circ (\tau \circ \delta) = \varphi \circ \beta$ for $\beta := \tau \circ \delta : Y \to M$, hence β exists as desired.

"⇐": Suppose g satisfies the condition. Want to show: f is irreducible. Since the given sequence does not split, f cannot be split mono or split epi. We have to show that f does not admit a non-trivial factorisation.

Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & \circlearrowleft & Y \\ & W \arrow[ur, "v"'] & \end{tikzcd}$$

be a factorisation of f.

f injective $\Rightarrow$ u injective $\Rightarrow \exists$ ses $0 \to X \xrightarrow{u} W \xrightarrow{\rho} \underset{\overset{\parallel}{M}}{\mathrm{Coker}(u)} \to 0$ and there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[r, "\rho"] \arrow[d, "v"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

where φ exists by the universal property of the cokernel.

Moreover, by Lemma 3.7, W is the pullback of

$$\begin{tikzcd} & M \arrow[d, "\varphi"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Now, φ has been chosen and by assumption, α or β exist.

First case: $\exists \alpha : M \to Y$ such that $\varphi = g \circ \alpha$ $\Rightarrow$ the following diagram commutes

$$\begin{tikzcd}
M \arrow[rrd, bend left, "Id_M"] \arrow[rd, dashed, "\exists \eta"] \arrow[rdd, bend right, "\alpha"'] & & \\
& W \arrow[r, "p"] \arrow[d, "v"'] \arrow[rd, phantom, "\ulcorner"] & M \arrow[d, "\varphi"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

$W =$ pullback $\Rightarrow \exists \eta : M \to W$ and $p \circ \eta = Id_M$ $\Rightarrow$ p split epi $\Rightarrow$ $0 \to X \xrightarrow{u} W \xrightarrow{p} M \to 0$ splits $\Rightarrow$ u is split mono, as desired

Second case: $\exists \beta : Y \to M$ such that $g = \varphi \circ \beta$ $\Rightarrow$ the following diagram commutes

$$\begin{tikzcd}
Y \arrow[rrd, bend left, "\beta"] \arrow[rd, dashed, "\exists \mu"] \arrow[rdd, bend right, "Id_Y"'] & & \\
& W \arrow[r, "p"] \arrow[d, "v"'] \arrow[rd, phantom, "\ulcorner"] & M \arrow[d, "\varphi"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

$W =$ pullback $\Rightarrow \exists \mu : Y \to W$ and $Id_Y = v \circ \mu$ $\Rightarrow$ v split epi, as desired

Altogether, f is irreducible $\square$

In Lemma 11.3 we have shown that a left almost split map has to start in an indecomposable module and a right almost split map has to end in an indecomposable module. For irreducible morphisms there is a similar result:

11.8 Lemma: (a) Let $f : X \to Y$ be an irreducible monomorphism. Then $Z := \operatorname{coker}(f)$ is indecomposable.

(b) Let $g : Y \to Z$ be an irreducible epimorphism. Then $X := \ker(g)$ is indecomposable

Proof: Again, the two assertions are related by duality. By way of a change, we are going to prove (b). Let $g : Y \to Z$ be an irreducible epimorphism and $X = \ker(g)$, $f : X \to Y$ the inclusion. Assume X decomposes non-trivially into $X = X_1 \oplus X_2$. Let $p_1 : X \to X_1$ and $p_2 : X \to X_2$ be the projections and consider

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "p_1"'] & Y \\
& X_1 &
\end{tikzcd}$$

Since g is irreducible, Proposition 11.7 (b) can be applied $\Rightarrow$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "p_1"'] & Y \arrow[dl, dashed, "\exists\alpha_1"] \arrow[dl, phantom, shift right=4ex, "\circlearrowleft"] \\
& X_1 &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "p_1"'] & Y \\
& X_1 \arrow[ur, dashed, "\exists\beta_1"'] &
\end{tikzcd}$$

But the second case is impossible: $f = \beta_1 \circ p_1$, f injective $\Rightarrow p_1$ injective ↯

Thus, α_1 must exist with $p_1 = \alpha_1 \circ f$. Analogously, $\exists \alpha_2 : Y \to X_2$ such that $p_2 = \alpha_2 \circ f$.

$id_X = (p_1, p_2) = (\alpha_1 \circ f, \alpha_2 \circ f) = (\alpha_1, \alpha_2) \circ f \Rightarrow f$ split mono $\Rightarrow g$ split epi ↯ □

Now we can make precise the connection between almost split morphisms and irreducible maps:

11.9 Theorem: (a) Let $f : X \to Y$ be left minimal almost split. Then:

(i) f is irreducible

(ii) Let $f_1 : X \to Y_1$ be a morphism also starting in X. Then f_1 is irreducible $\Leftrightarrow Y_1 \neq 0$ and Y has a decomposition $Y = Y_1 \oplus Y_2$ and there exists $f_2 : X \to Y_2$ such that $(f_1, f_2)^T : X \longrightarrow Y_1 \oplus Y_2$ is left minimal almost split.

(b) Let $g : Y \to Z$ be right minimal almost split. Then:

(i) g is irreducible

(ii) Let $g_1 : Y_1 \to Z$ be a morphism also ending in X. Then g_1 is irreducible $\Leftrightarrow Y_1 \neq 0$ and Y has a decomposition $Y = Y_1 \oplus Y_2$ and there exists $g_2 : Y_2 \to Z$ such that $(g_1, g_2) : Y_1 \oplus Y_2 \longrightarrow Z$ is right minimal almost split.

(The notation $(f_1, f_2)^T$ means $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.)

There are two messages in 11.9:

- Minimal almost split morphisms are irreducible.
- Irreducible morphisms starting or ending in an indecomposable module are direct summands of minimal almost split maps.

(Note: for all maps in 11.9 the domain X or the codomain Z must be indecomposable. Only Y can be decomposed.

If Y can be decomposed and it is the domain, then we are in situation (b), otherwise we are in situation (a). An irreducible map $U \overset{\alpha}{\to} V$ with U not injective and V not projective can occur both in situation (a) and in situation (b). Theorem 11.4 implies that α must occur in both situations.)

Why?

Proof of 11.9: We prove (a); (b) follows by duality.

Let $f: X \to Y$ be left minimal almost split.

(i) X is indecomposable by Lemma 11.3 (a).

f left minimal almost split $\Rightarrow$ f not split mono

Moreover, f not split epi: Otherwise, Y/X indecomposable $\Rightarrow$ f isomorphism $\Rightarrow$ split mono ↯

Want to show: any factorisation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & \circlearrowleft & Y \\ & Z \arrow[ur, "v"'] & \end{tikzcd}$$

is trivial.

If u_1 is split mono: done.

Otherwise:

f left minimal almost split $\Rightarrow$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "u"'] & Y \arrow[d, "\exists \alpha"] \\ & Z \end{tikzcd}$$

u factors as $u = \alpha \circ f$

$\Rightarrow f = v \circ u = v \circ \alpha \circ f$

f left minimal $\Rightarrow v \circ \alpha$ is an automorphism $\Rightarrow$ v split epi ✓

(Here we really have to use the minimality.)

(ii) "$\Rightarrow$" Let $f_1: X \to Y_1$ be irreducible. Then $Y_1 \neq 0$ and f_1 is not split mono.

$$\Rightarrow \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f_1"'] & Y \arrow[d, "\exists \alpha"] \\ & Y_1 \end{tikzcd}$$

f_1 factors as $f_1 = \alpha \circ f$

f_1 irreducible $\Rightarrow$ f split mono ↯ or α split epi

Since α must be split epi, $\exists$ isomorphism $\mu: Y \xrightarrow{\sim} Y_1 \oplus Y_2$ and $\mu = \binom{\alpha}{\beta}$ for some β.

f left minimal almost split, μ isomorphism $\Rightarrow \mu \circ f: X \to Y_1 \oplus Y_2$ also is left minimal almost split, and $\mu \circ f = \binom{\alpha}{\beta} \circ f = \binom{\alpha \circ f}{\beta \circ f} = \binom{f_1}{f_2}$ for $f_2 := \beta \circ f$. ✓

"$\Leftarrow$" Let $f_1: X \to Y_1$ be such a direct summand of $\binom{f_1}{f_2} = f: X \to Y_1 \oplus Y_2$ left minimal almost split.

Want to show: f_1 is irreducible.

$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is not split mono $\Rightarrow$ f_1 isn't either.

X indecomposable, f_1 not isomorphism $\Rightarrow$ f_1 not split epi

Let $$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[dr, "\alpha"'] & Y_1 \arrow[d, phantom, "\circlearrowleft"] \\ & M \arrow[u, "\beta"'] \end{tikzcd}$$ be a factorisation of f_1. If α is split mono: done ✓

Otherwise we have to show that β is split epi.

By (i), $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is irreducible. It has a factorisation $$\begin{tikzcd} X \arrow[rr, "\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} \alpha \\ f_2 \end{pmatrix}"'] & \arrow[d, phantom, "\circlearrowleft"] & Y_1 \oplus Y_2 \\ & M \oplus Y_2 \arrow[ur, "\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}"'] & \end{tikzcd}$$

$\Rightarrow$ $\begin{pmatrix} \alpha \\ f_2 \end{pmatrix}$ split mono or $\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$ split epi.

In the second case, β is split epi, too, and we are done.

We have to rule out the first case. Assume $\begin{pmatrix} \alpha \\ f_2 \end{pmatrix}$ is split mono.

X is indecomposable $\Rightarrow$ $\mathrm{End}_A(X)$ is a local ring. It has a unique maximal ideal, which coincides with the radical and

$$\mathrm{rad}(\mathrm{End}_A(X)) = \{ \varphi \in \mathrm{End}_A(X) : \varphi \text{ is not an isomorphism} \}$$

Consider the pre composition with α:

$$\mathrm{Hom}_A(M, X) \xrightarrow{-\circ\alpha} \mathrm{Hom}_A(X, X) = \mathrm{End}_A(X)$$

α not split mono $\Rightarrow$ $-\circ\alpha$ cannot be surjective, hence its image is contained in $\mathrm{rad}(\mathrm{End}_A(X))$.

$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ not split mono $\Rightarrow$ f_2 not split mono $\Rightarrow$ $-\circ f_2 : \mathrm{Hom}_A(Y_2, X) \to \mathrm{Hom}_A(X, X)$ not surjective and has image in $\mathrm{rad}\,\mathrm{End}_A(X)$

$\Rightarrow$ $\begin{pmatrix} \alpha \\ f_2 \end{pmatrix}$ - precomposing: $\mathrm{Hom}_A(M \oplus Y_2, X) \to \mathrm{End}_A(X)$ also has image in the radical

$\Rightarrow$ $\begin{pmatrix} \alpha \\ f_2 \end{pmatrix}$ cannot be split mono, and we have ruled out this possibility. $\square$

Example: Let Q be the Kronecker quiver $2 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 1$ and k any field. There are two indecomposable projective representations

$$P(1) = 0 \rightrightarrows k \quad \text{and} \quad P(2) = k \rightrightarrows k^2$$

$P(1)$ is simple, $\mathrm{rad}\,P(2) = 0 \rightrightarrows k^2 \simeq P(1) \oplus P(1)$ (with bases α and β, respectively). Thus, $\underset{\underset{0 \rightrightarrows k^2}{\shortparallel}}{P(1) \oplus P(1)} \hookrightarrow \underset{\underset{k \rightrightarrows k^2}{\shortparallel}}{P(2)}$ is right minimal almost split.

$P(1) \hookrightarrow P(2)$ is irreducible, and the right minimal almost split map

$e_1 \longmapsto$ linear combination of α and β

$\operatorname{rad} P(2) \hookrightarrow P(2)$ is a sum of two linearly independent irreducible maps in $\mathrm{Hom}_{kQ}(P(1), P(2))$.

Now we can state and prove the characterisation of almost split sequences that will be important in proving the existence theorem 11.4.

11.10 Theorem: Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in A-mod. Then the following assertions are equivalent.

(a) The sequence is almost split.

(b) X is indecomposable and g is right almost split.

(c) Z is indecomposable and f is left almost split.

(d) f is left minimal almost split.

(e) g is right minimal almost split.

(f) X and Z are indecomposable and f and g are irreducible.

(So, when proving 11.4 we can start with Z indecomposable non-projective, use the Auslander-Reiten formula to find an ses $0 \to D\,Tr\,Z \to E \to Z \to 0$, use 11.5 to see that $X = D\,Tr\,Z$ is indecomposable, as well, and then use 11.10 (b) to reduce the proof to checking that g is right almost split.)

Proof of 11.10: (a) ⇒ (d) and (a) ⇒ (e) are for free. Lemma 11.3 implies (a) ⇒ (b) and (a) ⇒ (c). Theorem 11.9. implies (a) ⇒ (f). Thus, (a) implies all other statements.

(e) ⇒ (b): By Theorem 11.9., g is irreducible, and by assumption g is surjective. Hence, by Lemma 9.8, $X \simeq \operatorname{Ker}(g)$ is indecomposable.

Similarly, (d) ⇒ (c).

We will show: (f) ⇒ (b), (b) ⇔ (c) and (b), (c) ⇒ (a).

(f) ⇒ (b): X is indecomposable by assumption, we have to show that g is right almost split. g irreducible ⇒ g not split epi. Let $\varphi: V \to Z$ be not split epi.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \\ & & & V \arrow[u, "\varphi"'] \arrow[ul, dashed, "\exists\alpha ?"] & \end{tikzcd}$$

Wlog V indecomposable. why?

g is assumed to be irreducible

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, dashed, "\beta"'] & Z \arrow[r] & 0 \\ & V \arrow[u, "\varphi"'] & \end{tikzcd}$$

⇒ By Proposition 11.7, φ factors as $g \circ \alpha$ or $\exists \beta$: such that g factors as $g = \varphi \circ \beta$. But g irreducible and φ is not split epi ⇒ β is split mono. However, V is indecomposable ⇒ β is an isomorphism and we can choose $\alpha = \beta^{-1}$.

(b) ⇔ (c): We show (b) ⇒ (c), the other implication is analogous.

By Lemma 11.3, Z is indecomposable. Want to show: f is left almost split.

f cannot be split mono, since g is not split epi.

Let $u: X \to U$ be any morphism.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[dl, dashed, "u'?"] \\ U & \end{tikzcd}$$

To show: u is split mono or $\exists u': u = u' \circ f$.

As usual, we form a pushout to get a commutative diagram with exact rows:

$$\begin{tikzcd} & 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"] \arrow[dl, phantom, "\lrcorner"] & Z \arrow[r] \arrow[d, equal] & 0 \\ (*) & 0 \arrow[r] & U \arrow[r, "h"'] & P \arrow[r, "\bar{g}"'] & Z \arrow[r] & 0 \end{tikzcd}$$

The ses (*) may or may not split.

First case: (*) splits ⇒ $\exists h': P \to U$ such that $h' \circ h = id_U$

Then $u = id_U \circ u = h' \circ h \circ u = h' \circ v \circ f = u' \circ f$ for $u' := h' \circ v: Y \to U$ ✓

Second case: (*) does not split. Then $\bar{g}$ is not split epi. g is right almost split

⇒ $\bar{g}$ factors:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, phantom, "\circlearrowright"] & Z \\ P \arrow[u, dashed, "\exists v'"] \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

$\bar{g} = g \circ v'$ ⇒ $g = \bar{g} \circ v = g \circ v' \circ v = \ldots$

$= g \circ (v' \circ v)^n \ \forall n \in \mathbb{N}$

$v' \circ v$ is an endomorphism of Y. Y may not be indecomposable, thus we cannot argue that $v' \circ v$ not nilpotent implies $v' \circ v$ is an isomorphism.

11.17

Since $j = g\circ v'$ we get: $g\circ v'\circ h = j\circ h = 0 \Rightarrow v'\circ h$ has to factor through the kernel of f, which is isomorphic to X with mono $f \Rightarrow \exists u' : U \to X$ such that

$$v'\circ h = f\circ u'$$

Then $\varphi := u'\circ u$ is an endomorphism of X, which is indecomposable and hence has local endomorphism ring

Entering everything in one big commutative diagram yields:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dd, bend right=60, "\varphi"'] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r, "h"'] \arrow[d, "u'"'] & P \arrow[r, "j"'] \arrow[d, "v'"'] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

$\text{End}_A(X)$ local

$\Rightarrow \varphi$ is nilpotent or an automorphism of X.

First case: φ is nilpotent, $\exists n : \varphi^n = 0$

$\Rightarrow \underset{\overset{\shortparallel}{0}}{\underline{f\circ 0}} = f\circ\varphi^n = (f\circ\varphi)\varphi^{n-1} = v'\circ v\circ f\circ\varphi^{n-1} = v'\circ v\circ f\circ\varphi\circ\varphi^{n-2} = (v'\circ v)^2\circ f\circ\varphi^{n-2} = \sim$

$= (v'\circ v)^n\circ f \Rightarrow (v'\circ v)^n$ factors through the cokernel of f, ie $Y \xrightarrow{g} Z$.

$\Rightarrow \exists\ \psi : Z \to Y$ such that $(v'\circ v)^n = \psi\circ g$. $\Rightarrow g = id_Z\circ g = g\circ(v'\circ v)^n = g\circ\psi\circ g$

g epi $\Rightarrow id_Z = g\circ\psi \Rightarrow g$ split epi $\lightning$

$\Rightarrow$ Only the second case is possible: $\varphi = u'\circ u$ is an automorphism of X

$\Rightarrow u$ is split mono ✓

(b), (c) $\Rightarrow$ (a): We only have to show that f and g are minimal. The two proof are analogous, we do the proof for f minimal.

Let $\alpha \in \text{End}_A(Y)$ such that $\alpha\circ f = f$. Want to show: α is an automorphism.

$g\circ\alpha\circ f = g\circ f = 0 \Rightarrow g\circ\alpha$ factors through the cokernel of f: $\exists\beta : Z \to Z$ such that $g\circ\alpha = \beta\circ g \rightsquigarrow$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "g"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r] & 0
\end{tikzcd}$$

Z is indecomposable $\Rightarrow \mathrm{End}_A(Z)$ is local $\Rightarrow \beta$ is nilpotent or an automorphism

First case: β is nilpotent, $\exists n: \beta^n = 0 \Rightarrow 0 = 0 \circ g = \beta^n \circ g = \beta^{n-1} \circ \beta \circ g =$
$= \beta^{n-1} \circ g \circ \alpha = \text{–} = g \circ \alpha^n \Rightarrow \alpha^n$ factors through the kernel of g

$\Rightarrow \exists \mu: Y \to X$ such that $\alpha^n = f \circ \mu$

$\Rightarrow f = f \circ id_X = \alpha \circ f = \text{–} = \alpha^n \circ f = f \circ \mu \circ f \Rightarrow \mu \circ f = id_X$, since f is mono

$\Rightarrow f$ split mono ↯

Thus we always are in the second case: β is an automorphism. The short five Lemma 1A.1 implies that α is an automorphism. ✓ □

This finishes the long step (b) in the proof of 11.4. Now we do the easy part (c), proving uniqueness.

11.11 Lemma: (a) Let $f_1: X \to Y_1$ and $f_2: X \to Y_2$ be left minimal almost split. Then there is an isomorphism $\varphi: Y_1 \to Y_2$ such that

$$\begin{tikzcd} & Y_1 \arrow[dd, "\wr"', "\varphi"] \\ X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] \arrow[r, phantom, "\circlearrowleft"] & \\ & Y_2 \end{tikzcd}$$

commutes.

(b) Let $g_1: Y_1 \to Z$ and $g_2: Y_2 \to Z$ be right minimal almost split. Then there is an isomorphism ψ such that

$$\begin{tikzcd} Y_1 \arrow[dd, "\wr"', "\psi"] \arrow[dr, "g_1"] & \\ \arrow[r, phantom, "\circlearrowleft"] & Z \\ Y_2 \arrow[ur, "g_2"'] & \end{tikzcd}$$

commutes.

Proof: (b) is dual to (a), we prove (a).

f_1 left almost split and f_2 not split mono $\Rightarrow \exists \varphi$:

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[dr, "f_2"'] \arrow[dr, phantom, "\circlearrowleft", near start, shift left=3] & Y_1 \arrow[d, "\varphi"] \\ & Y_2 \end{tikzcd}$$

f_2 left almost split and f_1 not split mono

$\Rightarrow \exists \varphi'$:

$$\begin{tikzcd} & Y_1 \\ X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] \arrow[r, phantom, "\circlearrowleft"] & \\ & Y_2 \arrow[uu, "\varphi'"'] \end{tikzcd}$$

So: $f_2 = \varphi \circ f_1 = \varphi \circ \varphi' \circ f_2$ and $f_1 = \varphi' \circ \varphi \circ f_1$

f_1 and f_2 are left minimal $\Rightarrow \varphi \circ \varphi' \in \mathrm{End}_A(Y_2)$ and $\varphi' \circ \varphi \in \mathrm{End}_A(Y_1)$ both are automorphisms. This implies φ and φ' are split monos and split epis, hence isomorphisms. □

So, there is at most one left minimal almost split morphism leaving X, up to isomorphism, and at most one right minimal almost split morphism arriving at Z, up to isomorphism.

Given two almost split sequences starting in X, Lemma 11.11 provides an isomorphism $\varphi: Y_1 \simeq Y_2$, which induces an isomorphism $\tau: Z_1 \to Z_2$ of cokernels, and altogether an isomorphism of the two almost split sequences.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d, "\wr\ \varphi"', "\wr"] & Z_1 \arrow[r] \arrow[d, "\tau"', "\wr"] & 0 \\
0 \arrow[r] & X \arrow[r] & Y_2 \arrow[r] & Z_2 \arrow[r] & 0
\end{tikzcd}$$

This establishes the uniqueness of almost split sequences claimed in Theorem 11.4.

The most interesting part of 11.4 is existence. What remains to prove existence is to combine Chapter 10, the Auslander-Reiten formulae, with Theorem 11.10. This will be the proof of 11.4 (or rather step (d)) to be given next. This is also the most interesting part of the proof.