Representation theory of the Kronecker algebra

Let $k=\bar{k}$, $A=kQ$ where $Q = \cdot \rightrightarrows \cdot$ ($=Q^{op}$, so we may consider left or right modules).
First we compute the Cartan matrix $C_A = \begin{pmatrix}1&2\\0&1\end{pmatrix}$, $C_A^{-1} = \begin{pmatrix}1&-2\\0&1\end{pmatrix}$ and the Coxeter matrix $\Phi_A = \begin{pmatrix}-1&2\\-2&3\end{pmatrix} = \underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{= I} + \underbrace{\begin{pmatrix}-2&2\\-2&2\end{pmatrix}}_{= N}$. $N^2=0 \Rightarrow \Phi_A^{-1} = I-N$ and $\Phi_A^{n} = I+nN$ for $n\in\mathbb{Z}$.

$\rightsquigarrow$ The preprojective component $\mathcal{P}$ contains the indecomposable modules with dimension vector $\underline{\dim}\, P_1 = \binom{1}{0}$, $\underline{\dim}\, \tau^{-n}P_1 = \binom{1}{0}+\binom{2n}{2n}$ for $n\in\mathbb{N}$

$\underline{\dim}\, P_2 = \binom{2}{1}$, $\underline{\dim}\, \tau^{-n}P_2 = \binom{2}{1}+\binom{2n}{2n}$ for $n\in\mathbb{N}$ $\qquad$ $\underline{\dim} = \binom{a}{b}$, $a-b=1$

and the preinjective component $\mathcal{I}$ contains indecomposables with

$\underline{\dim}\, I_1 = \binom{1}{2}$, $\underline{\dim}\, \tau^{n}I_1 = \binom{1}{2}+\binom{2n}{2n}$ for $n\in\mathbb{N}$

$\underline{\dim}\, I_2 = \binom{0}{1}$, $\underline{\dim}\, \tau^{n}I_2 = \binom{0}{1}+\binom{2n}{2n}$ for $n\in\mathbb{N}$ $\qquad$ $\underline{\dim} = \binom{a}{b}$, $a-b=-1$

Exercise (a): Draw the Auslander-Reiten components $\mathcal{P}$ and $\mathcal{I}$. Knit them.

Let X be indecomposable, $X\notin\mathcal{P}$, $X\notin\mathcal{I}$, $\underline{\dim}\, X = \binom{a}{b}$. Then all $\tau^{\pm n}X$ exist and $\Phi_A^{\pm n}$ sends $\binom{a}{b}$ to dimension vectors, i.e. nonnegative entries and not the zero vector. This implies $a=b$. Regular modules have dimension vectors $\binom{n}{n}$ for $n\in\mathbb{N}$.

Exercise (b): Let X and Y be regular (not necessarily indecomposable) and $f\in \mathrm{Hom}_A(X,Y)$. Show that $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ and $\mathrm{Coker}(f)$ are regular, too.
Deduce that $\mathrm{add}\,\mathcal{R}$ (the category whose objects are finite direct sums of indecomposable regular modules) is an abelian category.

(This is an abelian category without any non-zero projective or injective object.)

Let $\mathbb{P}^1(k)$ be the projective line over k, that is, the set of all one-dimensional subspaces of $k \oplus k$, represented by basis vectors. $\begin{bmatrix}\lambda \\ \mu\end{bmatrix}$ is the subspace generated by $\begin{pmatrix}\lambda \\ \mu\end{pmatrix}$.

For $p = \begin{bmatrix}\lambda \\ \mu\end{bmatrix}$ let R_p be $k \overset{\lambda}{\underset{\mu}{\rightrightarrows}} k$. R_p is independent of the choice of $\begin{pmatrix}\lambda \\ \mu\end{pmatrix}$.

Exercise (c): Check that for $\begin{bmatrix}\lambda \\ \mu\end{bmatrix} = p$, R_p is indecomposable and regular and

$$\mathrm{End}_A(R_p) = k$$

$$\mathrm{Hom}_A(R_p, R_q) = \begin{cases} k & \text{for } p = q \text{ (i.e. same subspace generated)} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^1_A(R_p, R_p) = k$$

Exercise (d): Show that every regular module has a submodule R_p for some p.

Exercise (e): Show that $\tau(R_p) \cong R_p$ and determine the almost split sequence ending in R_p.

Exercise (f): Knit the AR component containing R_p and draw it on a semi-infinite cylindre.

Exercise (g): Show that the component determined in (f), usually called a tube, yields an abelian category in itself (when allowing to form finite direct sums).

Exercise (h): Prove that $\mathcal{R}$ is the disjoint union (and as category the direct sum) of the tubes (one for each p).