

Auslander-Reiten quivers of path algebras

Let Q be a connected quiver without loops or oriented cycles, and $A := kQ$.

Recall: A is hereditary and there are no morphisms from non-projective to projective or injective to non-injective in decomposable modules.

Exercise (a): Let X, Y be indecomposable, not projective. Show that

$$\text{Hom}_A(X, Y) = \underline{\text{Hom}}_A(X, Y)$$

$$\text{and } \text{Hom}_A(X, Y) \cong \text{Hom}_A(\tau X, \tau Y)$$

(and similarly for Hom and τ^{-1}).

An indecomposable A -module M is called

preprojective : $\Leftrightarrow \exists P$ indecomposable projective, $n \in \mathbb{N}_0$ such that $M = \tau^{-n} P$

preinjective : $\Leftrightarrow \exists I$ indecomposable injective, $n \in \mathbb{N}_0$ such that $M = \tau^n I$

regular : $\Leftrightarrow M$ is neither preprojective nor preinjective

Exercise (b): Show that all indecomposable projective modules are in the same AR component, and similarly for injective.

Exercise (c): Prove that the preprojective modules are in the same ^{AR-} component, and this AR-component only contains preprojective modules. Similarly for preinjective. Deduce that A has finite representation type if and only if all modules are preprojective.

Suppose A has infinite representation type. Let \mathcal{P} be the preprojective component, \mathcal{I} the preinjective and \mathcal{R} the union of the regular components.

Exercise (d): Prove that $X \notin \mathcal{P}, Y \in \mathcal{P} \Rightarrow \text{Hom}_A(X, Y) = 0$

$$X \in \mathcal{I}, Y \notin \mathcal{I} \Rightarrow \text{Hom}_A(X, Y) = 0$$

$$\mathcal{P} \cap \mathcal{I} = \emptyset$$