The Coxeter transformation

To compute $D\,Tr\,M = \tau M$ or $Tr\,D\,M = \tau^{-1}M$ for M indecomposable (and not projective or not injective, respectively) can be difficult and it is also not easy to see if one made a mistake. The dimension vector $\underline{\dim}\, M$ is a combinatorial shadow of M (and sometimes determines it up to isomorphism). Therefore we ask for $\underline{\dim}\,\tau M$ and $\underline{\dim}\,\tau^{-1}M$. Can we compute these from $\underline{\dim}\, M$?

Recall definition 7.5: Let A be a basic algebra with simples $S_1, \dots, S_n$ and $_AA = P_1 \oplus \dots \oplus P_n$.
Then the Cartan matrix C_A has i-th column $\left.\begin{matrix}[P_i : S_1] \\ \vdots \\ [P_i : S_n]\end{matrix}\right\}$ composition multiplicities of P_i (well-defined by the theorem of Jordan-Hölder).
For $A = kQ/I$ a bound quiver algebra, the columns of C_A are exactly the dimension vectors $\underline{\dim}\, P_1, \dots, \underline{\dim}\, P_n$.
By exercise 7.6, $gl\dim A < \infty$ implies $\det C_A = \pm 1$, ie C_A is invertible over $\mathbb{Z}$.

Let $gl\dim A < \infty$. The Coxeter matrix of A is $\Phi_A := -\underbrace{C_A^T}_{\text{transposed}} C_A^{-1}$. The linear transformation $\Phi_A : \mathbb{Z}^n \to \mathbb{Z}^n$ is called Coxeter transformation.
Examples: $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} = k(\cdot \to \cdot)$: $C_A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $C_A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $C_A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\Phi_A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$
$A = k(\cdot \rightrightarrows \cdot)$ Kronecker algebra.
Then $C_A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $C_A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, $C_A^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\Phi_A = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$

Exercise (a): Show that $\underline{\dim}\, P(j) = C_A\, \underline{\dim}\, S(j)$, $(\underline{\dim}\, I(j))^T = (\underline{\dim}\, S(j))^T C_A$ and
$$\underline{\dim}\, I(j) = -\Phi_A\, \underline{\dim}\, P(j)$$
(which may help to compute Φ_A directly)

Notation: $\nu := \underbrace{D\,Hom_A(-, A)}_{= D(-)^t} : A\text{-mod} \to A\text{-mod}$ is the Nakayama functor, ie. $D\,Tr\,M$ is the kernel of $\nu(p_1)$ in the definition of $D\,Tr$.

Given M indecomposable, not projective, then

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0 \quad \text{(minimal projective presentation)}$$

$$\rightsquigarrow \quad 0 \to M^t \xrightarrow{p_0^t} P_0^t \xrightarrow{p_1^t} P_1^t \to \mathrm{Tr}\,M \to 0$$

$$\rightsquigarrow \quad 0 \to D\mathrm{Tr}\,M \to \nu(P_1) \xrightarrow{\nu(p_1)} \nu(P_0) \xrightarrow{\nu(p_0)} \nu(M) \to 0$$

Exercise (b): Show that $\underline{\dim}\,\tau X = \Phi_A(\underline{\dim}\,X) - \Phi_A(\underline{\dim}\,\mathrm{Ker}(p_1)) + \underline{\dim}\,\nu X$
(and dually, from $0 \to Y \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1$:

$$\underline{\dim}\,\tau^{-1} Y = \Phi_A^{-1}(\underline{\dim}\,Y) - \Phi_A^{-1}(\underline{\dim}\,\mathrm{Coker}(i_1)) + \underline{\dim}\,(DY)^t$$

(Here, X is indecomposable not projective and Y is indecomposable not injective.)

These formulae are complicated. When X has projective dimension one, then $0 \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X \to 0$ has p_1 injective and $\mathrm{Ker}(p_1) = 0$. Hence, on the right hand side there are only two terms left.
When $\mathrm{gl\,dim}\,A \leq 1$ (eg $A = kQ$), $\mathrm{Ker}(p_1) = 0$ and $\mathrm{Coker}(i_1) = 0$ happens all the time). Such an algebra is called hereditary. Example: $A = kQ$, the path algebra of a quiver.

Exercise (c): Check that A (finite dimensional) is hereditary
$\Leftrightarrow$ submodules of projective modules are projective
$\Leftrightarrow$ quotient modules of injective modules are injective
(Thus, being projective is inherited by submodules.)

Exercise (d): Let A be hereditary. Show that for X indecomposable not projective $\mathrm{Hom}_A(X, A) = 0$. Hence $\underline{\dim}\,\tau X = \Phi_A(\underline{\dim}\,X)$ and for Y indecomposable not injective: $\underline{\dim}\,\tau^{-1} Y = \Phi_A^{-1}(\underline{\dim}\,X)$.

(Here, only dimension vectors, Φ_A and Φ_A^{-1} are needed, and nothing else.)

Exercise (e): Compute the Auslander Reiten quivers of $k(\to\to)$, $k(\to\leftarrow)$ and $k(\leftarrow\to)$.