Are indecomposable modules determined by their dimension vectors?

For $A = kQ/I$, a representation $V = (V(i)_{i \in Q_0}, V(\alpha)_{\alpha \in Q_1})$ has dimension vector $\underline{\dim} V = (\dim_k V(1), \dim_k V(2), \dots)$.
For A any finite dimensional algebra with simple modules $S(1), \dots, S(n)$ (up to isomorphism), the dimension vector $\underline{\dim} V = (d_1, \dots, d_n)$ of an A-module V records the multiplicities d_i of $S(i)$ in any composition series of V (these are uniquely defined by the theorem of Jordan and Hoelder).

Naive question: Is V determined by $\underline{\dim} V$? More precisely, does $\underline{\dim} V = \underline{\dim} W$ imply $V \simeq W$?
Answer: For any vector $\underline{d} = (d_1, \dots, d_n)$, the module $\underbrace{S(1) \oplus \dots \oplus S(1)}_{d_1} \oplus \underbrace{S(2) \oplus \dots}_{d_2} \dots$ has dimension vector $\underline{d}$ $\Rightarrow$ Only for A semisimple the answer is yes.
So we should ask a less naive question:
Question: Let V and W be indecomposable A-modules such that $\underline{\dim} V = \underline{\dim} W$. Can we conclude that V and W are isomorphic?

The answer cannot be positive in general: When Brauer-Thrall II is satisfied, there exists $m \in \mathbb{N}$ such that there are infinitely many pairwise non-isomorphic m-dimensional A-modules. Infinitely many of these must have the same dimension vector. So, the question should be asked for A of finite representation type. But the answer still can be negative:

(a) Consider $A = kQ/I$ where $Q = \tilde{A}_2 = 1 \rightleftarrows 2$. Show that in most cases (i.e. for most choices of I) there exist indecomposable V and W, $V \ncong W$ such that $\underline{\dim} V = \underline{\dim} W$.

This is not a reason to give up. When we compute the Auslander Reiten quivers of $A = kQ$ for $Q = A_n$ (any orientation), we see that the question has a positive answer. This implies a positive answer also for all $A = kQ/I$, I of type A_n.
Thus, the problem really is to find an assumption on A or A-mod that implies a positive answer (for all indecomposable modules or for particular classes of such).

<u>Answer:</u> Let A have finite representation type and assume there is no cycle of irreducible maps in A-mod. Then the indecomposable A-modules are uniquely determined by their composition factors.
(Here, a cycle of irreducible maps means: $X_1 \to X_2 \to \cdots \to X_e \to X_1$ where all X_i are indecomposable, all maps are irreducible.
The composition of the maps is allowed to be zero.)
This is a theorem of Happel and Ringel. The proof in the following exercises is due to Happel.

(b) Let $0 \to DTrY \to E \to Y \to 0$ be an almost split sequence and M indecomposable and $M \not\cong DTrY$. Then $Hom_A(-,M)$ yields an exact sequence

$$0 \to Hom_A(Y,M) \to Hom_A(E,M) \to Hom_A(DTrY,M) \to 0$$

Now we fix indecomposable A-modules M and N with $\underline{dim}\,M = \underline{dim}\,N$. We assume $M \not\cong N$ and aim at a contradiction. We also fix minimal left almost split maps $M \to B = \bigoplus_{i=1}^{r} B_i$ and $N \to C = \bigoplus_{j=1}^{s} C_j$, all B_i, C_j indecomposable.
Set $\mathcal{Y} := \{Y \text{ indecomposable}: \exists \text{ finite chain of irreducible maps } Y \to \dots \to B_i, \text{ some } i\}$
and $\mathcal{Z} := \{Z \text{ indecomposable}: \quad \text{—"—} \quad Z \to \dots \to C_j, \text{ some } j\}$,
where the chains run through indecomposable modules.

(c) Show that $TrDM \notin \mathcal{Y}$, $TrDN \notin \mathcal{Z}$ and $[TrDN \notin \mathcal{Y}$ or $TrDM \notin \mathcal{Z}]$

Without loss of generality we assume from now on $\operatorname{Tr} D N \notin \mathcal{Y}$.

For any indecomposable module X we define

$$h(X) := \sup\{\ell : \exists \text{ chain of irreducible maps between indecomposables } P = X_\ell \to X_{\ell-1} \to \dots \to X_1 \to X_0 = X,\ P \text{ projective}\} \in \mathbb{N}_0 \cup \{\infty\}$$

(d) Show that $h(X) \in \mathbb{N}_0$ $\forall X$.

(e) Show by induction on $h(Y)$ that $\dim_k \operatorname{Hom}_A(Y,M) = \dim_k \operatorname{Hom}_A(Y,N)$ $\forall Y \in \mathcal{Y}$.

(f) Derive a contradiction to the assumption that $M \ncong N$.

(A variation on this proof also works for certain infinite components of Auslander Reiten quivers, called preprojective or preinjective components.)