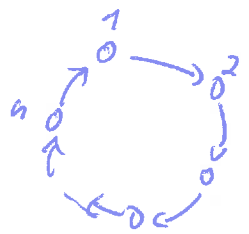
Exercises on almost split sequences (1)

Let Q be one of the following quivers (for $n \geq 1$)

$$\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \circ \longrightarrow \cdots \longrightarrow \overset{n}{\circ} \quad (A_n, \text{ ordered from the left to the right})$$



($\tilde{A}_n$, with cyclic order)

Let k be a field and choose I such that the bound quiver algebra $A = kQ/I$ is finite dimensional and has n simple modules, up to isomorphism.

(a) Check that the indecomposable projective modules $P(1), \dots, P(n)$ are uniserial, that is, they have unique composition series.
Check that all $\operatorname{rad}^t P(i)$ and $P(i)/\operatorname{rad}^t P(i)$ are uniserial, too, if non-zero.

(b) Suppose $Z := P(i)/\operatorname{rad}^t P(i)$ is non-zero and non-projective. Show that in the almost split sequence $0 \to X \to Y \to Z \to 0$, the terms X and Y satisfy

$$X \simeq \operatorname{rad} P(i)/\operatorname{rad}^{t+1} P(i)$$

and $$Y \simeq P(i)/\operatorname{rad}^{t+1} P(i) \oplus \frac{\operatorname{rad} P(i)}{\operatorname{rad}^t P(i)}$$ (omit the second summand if zero)

(Theorem 11.10 suggests different approaches to this problem.)

(c) What happens in (b) when Z is simple?

(d) Does X in (b), if non-projective, also have the form $Z = P(\ell)/\operatorname{rad}^h P(\ell)$ for some h, ℓ, so that one can continue forming almost split sequences?