§10. Auslander's defect formula and the Auslander–Reiten formulae

Let A be a finite dimensional algebra over a field K, ${}_AP$ projective and $0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0$ a short exact sequence of A-modules.
By Theorem 3.3, applying $\mathrm{Hom}_A(-,X)$ for $X \in A$-mod yields an exact sequence
$0 \to \mathrm{Hom}_A(M,X) \xrightarrow{g^*} \mathrm{Hom}_A(P,X) \xrightarrow{f^*} \mathrm{Hom}_A(K,X) \to \mathrm{Ext}^1_A(M,X) \to 0$
So, $\mathrm{Ext}^1_A(M,X)$ is isomorphic to the cokernel of f^*.
When the middle term P is not required to be projective, the situation changes, as we know from Theorem 3.10:
Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be exact and N any module. Then there is an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(Z,N) \arrow[r, "g^*"] & \mathrm{Hom}_A(Y,N) \arrow[r, "f^*"] & \mathrm{Hom}_A(X,N) \arrow[dll, "\delta"] \\
& \mathrm{Ext}^1_A(Z,N) \arrow[r] & \mathrm{Ext}^1_A(Y,N) \arrow[r] & \mathrm{Ext}^1_A(X,N) \arrow[r] & {}
\end{tikzcd}
$$

(to be continued by $\mathrm{Ext}^2_{(-)}$ as we have learnt in Chapter 6).
Although $\mathrm{Coker} f^*$ in general does not tell us what $\mathrm{Ext}^1_A(Z,N)$ is, it may contain interesting information.
We know already: $\mathrm{Coker} f^* = 0 \Leftrightarrow \mathrm{Hom}_A(-,N)$ is exact on $0 \to X \to Y \to Z \to 0$ and N injective implies that.
But we don't know what to expect when $\mathrm{Coker} f^* \neq 0$.
<u>Question:</u> What is the meaning of $\mathrm{Coker} f^*$?

Important: $\mathrm{Coker} f^*$ is functorial in the variable N: Let $\varphi: N \to N'$

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(Z,N) \arrow[r] \arrow[d, "\varphi_*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(Y,N) \arrow[r] \arrow[d, "\varphi_*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(X,N) \arrow[r] \arrow[d, "\varphi_*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Coker} f^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(Z,N') \arrow[r] & \mathrm{Hom}_A(Y,N') \arrow[r] & \mathrm{Hom}_A(X,N') \arrow[r] & \mathrm{Coker} f^* \arrow[r] & 0
\end{tikzcd}
$$

Here, φ_* means postcomposing with φ and the morphism between the cokernels is induced by φ_*, which follows from the universal property of the cokernel, or by a diagram chase.

$\rightsquigarrow$ This defines a covariant functor δ_* such that

$$0 \to \mathrm{Hom}_A(Z,-) \xrightarrow{g^*} \mathrm{Hom}_A(Y,-) \to \mathrm{Hom}_A(X,-) \to \delta_* \to 0 \text{ is exact.}$$

Similarly, there is an exact sequence of functors

$$0 \to \mathrm{Hom}_A(-,X) \xrightarrow{f_*} \mathrm{Hom}_A(-,Y) \xrightarrow{g_*} \mathrm{Hom}_A(-,Z) \to \delta^* \to 0$$

where the contravariant functor δ^* is defined by $\mathrm{Coker}\, g_*$.
(That the sequence of functors is exact means that it is exact for any module we can plug in.)

<u>10.1 Definition:</u> The functor $\delta_*: A\text{-mod} \to k\text{-Vect}$ is called the <u>covariant defect</u>.
The functor $\delta^*: A\text{-mod} \to k\text{-Vect}$ is called the <u>contravariant defect</u>.

As the names indicate, δ_* and δ^* measure how far away from being exact the sequences of Hom-functors are.
Both defect functors δ_* and δ^* are defined using the same ses $0 \to X \to Y \to Z \to 0$
<u>Question:</u> How are δ_* and δ^* related to each other?

How should one approach such a question?
We know: $\mathrm{Hom}_A(P,-)$ is exact $\Leftrightarrow$ P projective, hence $\delta^*(P) = 0$ for P projective.
And $\delta_*(I) = 0$ for I injective.
Recall: The A-projectives are add(A), the direct summands (up to isomorphism) of the $A^n = \underbrace{A \oplus \dots \oplus A}_{n \text{ summands}}$ for $n \in \mathbb{N}$. The A-injectives are add$(D(A_A))$, where A_A is the regular right module and $D = \mathrm{Hom}_k(-,k)$ is vectorspace duality, sending a k-vector space V to the dual space V^* and a linear map $\alpha: V \to W$ to the dual map $\alpha^*: W^* \to V^*$, which in terms of the dual bases is given by the transposed matrix.
k-duality D turns a short exact sequence $0 \to X \to Y \to Z \to 0$ into $0 \to DZ \to DY \to DX \to 0$, which may help to move between covariant and contravariant.

In the case of projective modules there is another way to move from left to right modules and back: Indecomposable projective left A-modules are isomorphic to some Ae, $e=e^2$ a primitive idempotent.

$$\mathrm{Hom}_A(Ae, A) \xrightarrow{\sim} eA \quad (\text{indecomposable projective right module})$$
$$(\alpha : Ae \to A) \longmapsto \alpha(e)$$

and $\underset{\substack{\uparrow \\ \text{right module} \\ \text{homomorphisms}}}{\mathrm{Hom}_{A^{op}}}(eA, A) \xrightarrow{\sim} Ae \Rightarrow \mathrm{Hom}_A(-,A) : \underset{(\text{left})}{A\text{-proj}} \longrightarrow \underset{(\text{right})}{\text{proj-}A}$ gives a bijection

(but not an equivalence of categories:
$\mathrm{Hom}_A(Ae, Af) = eAf$ while $\mathrm{Hom}_{A^{op}}(eA, fA) = fAe$
$\| \ \mathrm{Hom}_A(Af, Ae)$)

For P projective, ${}_AP \simeq \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(P,A), A_A)$,
for a general module ${}_AX$, this does not work.
However, ${}_AM \in A$-mod has a projective presentation $P_1 \xrightarrow{\alpha} P_0 \to M \to 0$ with P_0, P_1 projective.
= first two steps of a projective resolution
Abbreviating $* = \mathrm{Hom}_A(-,A)$ this yields

$$0 \to M^* \longrightarrow P_0^* \xrightarrow{\alpha^*} P_1^* \qquad \begin{matrix} (Ae)^* & (Af)^* \\ \| & \| \end{matrix}$$

(where α^* corresponds to α via $\mathrm{Hom}_A(Af, Ae) = fAe = \mathrm{Hom}_{A^{op}}(eA, fA)$)

10.2 Definition: The cokernel of $\alpha^* : P_0^* \longrightarrow P_1^*$ is called the transpose of M.
Notation: $\mathrm{Tr}\,M$.

In order to avoid confusion we use DV for dual vectorspaces and $D\alpha$ for dual maps, and P^* and α^* when applying $\mathrm{Hom}_A(-,A)$.

(Auslander's defect formula)
10.3 Theorem: Let $\delta : 0 \to X \to Y \to Z \to 0$ be an exact sequence of A-modules.
Then $D\delta^*(M) \simeq \delta_*(D\,\mathrm{Tr}\,M)$ for A-modules M.
The isomorphism is functorial in the exact sequences δ and in the modules M.

So, on the left hand side we apply $\mathrm{Hom}_A(M,-)$ to the ses δ, take the cokernel and then the dual vector space. On the right hand side we apply $\mathrm{Hom}_A(-, D\,\mathrm{Tr}\,M)$ and take the cokernel.

By definitions, the left hand side is the vectorspace dual of a quotient of $\mathrm{Hom}_A(M,Z)$. The right hand side is a quotient of $\mathrm{Hom}_A(X, D\,\mathrm{Tr}M)$. There is no obvious map between these Hom-spaces.

Proof of 10.3: Fix $\delta: 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ and M and choose a projective presentation $P_1 \xrightarrow{\alpha} P_0 \to M \to 0$ of M. By definition of $\mathrm{Tr}M$, $P_0^* \xrightarrow{\alpha^*} P_1^* \to \mathrm{Tr}M \to 0$ is exact. $\mathrm{Tr}M$, P_0^* and P_1^* are right A-modules.

Apply $\mathrm{Hom}_A(-,X)$, which is left exact, and $-\otimes_A X$, which is right exact, to get

$$\begin{tikzcd}
 & P_0^* \otimes_A X \arrow[r] \arrow[d, dashed, "\exists\beta ?"'] \arrow[dr, phantom, "\circlearrowleft"] & P_1^* \otimes_A X \arrow[r] \arrow[d, dashed, "? \exists\gamma"] & \mathrm{Tr}M \otimes_A X \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(M,X) \arrow[r] & \mathrm{Hom}_A(P_0,X) \arrow[r] & \mathrm{Hom}_A(P_1,X) &
\end{tikzcd}$$

why is this exact?

why is this exact?

Claim: there exist isomorphisms β and γ making the square commutative.

On a direct summand Ae of P_0: $\mathrm{Hom}_A(P_0,X)$ has the summand

$$\mathrm{Hom}_A(Ae,X) \xrightarrow{\simeq} eX$$

and $P_0^* \otimes X$ has the summand $\mathrm{Hom}_A(Ae,A) \otimes_A X = eA \otimes_A X \simeq eX$

This defines isomorphisms β and γ.

Comparing α and α^* shows commutativity. why? $\Rightarrow$ Claim ✓

Identifying now terms that are isomorphic by β or γ, respectively, yields the exact sequence $0 \to \mathrm{Hom}_A(M,X) \to \mathrm{Hom}_A(P_0,X) \to \mathrm{Hom}_A(P_1,X) \to \mathrm{Tr}M \otimes_A X \to 0$ and analogous sequences with X replaced by Y or Z, respectively.

$\Rightarrow$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}_A(M,X) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_0,X) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_1,X) \arrow[r] \arrow[d] & \mathrm{Tr}M \otimes_A X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(M,Y) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_0,Y) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_1,Y) \arrow[r] \arrow[d] & \mathrm{Tr}M \otimes_A Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(M,Z) \arrow[r] & \mathrm{Hom}_A(P_0,Z) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_1,Z) \arrow[r] \arrow[d] & \mathrm{Tr}M \otimes_A Z \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

This diagram is commutative. The rows are exact, as we have seen. The columns are exact, since Hom is left exact, tensoring is right exact and since P_0 and P_1 are projective. The Snake Lemma 1A.3 provides the connecting homomorphism (in black): $\operatorname{Hom}_A(M,Z) \to Tr M \otimes_A X$.

$$\Rightarrow \delta^*(M) \overset{\text{by def}}{=} \operatorname{Coker}(\operatorname{Hom}_A(M,Y) \to \operatorname{Hom}_A(M,Z))$$

$$\simeq \operatorname{Ker}(Tr M \otimes_A X \longrightarrow Tr M \otimes_A Y)$$ by exactness of the sequence in the Snake Lemma

$$\Rightarrow D\delta^*(M) \simeq D \operatorname{Ker}(Tr M \otimes_A X \longrightarrow Tr M \otimes_A Y)$$

$$\simeq \operatorname{Coker} D(Tr M \otimes_A X \longrightarrow Tr M \otimes Y)$$ since for a map $h: V \to W$, $D\operatorname{Ker}(h) \simeq \operatorname{Coker} Dh$

$$\simeq \operatorname{Coker}(\operatorname{Hom}_A(Y, DTrM) \to \operatorname{Hom}_A(X, DTrM))$$

since $D(TrM \otimes Y) = \operatorname{Hom}_k(TrM \otimes_A Y, k) \simeq \operatorname{Hom}_A(Y, DTrM)$ by Theorem 8.9 (adjoint isomorphisms)

$$\simeq \delta_*(DTrM)$$

This proves the formula.

To check functoriality, one can use functoriality of Hom, tensor, D and $*$. The only problem is DTr.

Claim: given $h: M \to M'$, there is a unique map, induced by h,

$$\delta_*(DTrM) \to \delta_*(DTrM')$$

Given $P_1 \xrightarrow{\alpha} P_0 \to M \to 0$ and $Q_1 \xrightarrow{\beta} Q_0 \to M' \to 0$ projective presentations (not unique), h induces

$$\begin{tikzcd}
0 \arrow[r] & DTrM \arrow[r] \arrow[d, "\hat{h}"'] \arrow[dr, phantom, "\circlearrowleft"] & DP_1^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & DP_0^* \arrow[d] \\
0 \arrow[r] & DTrM' \arrow[r] & DQ_1^* \arrow[r] & DQ_0^*
\end{tikzcd}$$

$\hat{h}$ is not unique, but two choices for $\hat{h}$ only differ by a morphism factoring through DP_1^*, which is an injective left A-module. δ_* vanishes on injective modules $\Rightarrow \delta_*(\hat{h})$ is unique. □

When M is projective, hence $\delta^*(M) = 0$, hence $DTrM$ is injective, which also follows from the construction. So, $DTrM$ is different from M, in general. This will turn out to be crucial for constructing new modules from given ones.

As direct consequences of the defect formula, we get new connections between Hom and Ext^1:

Notation: Let M and N be A-modules. Let $P(M,N) := \{ f : M \to N,$ f A-homomorphism, $\exists P$ projective such that f factors $\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N \\ & P \arrow[ur] & \end{tikzcd}$ $\}$

$P(M,N)$ is a subspace of $\mathrm{Hom}_A(M,N)$

$\underline{\mathrm{Hom}}_A(M,N) := \mathrm{Hom}_A(M,N) / P(M,N)$ (quotient space).

Dually, $I(M,N)$ are the morphisms factoring through an injective module.

$\overline{\mathrm{Hom}}_A(M,N) := \mathrm{Hom}_A(M,N) / I(M,N)$

<u>10.4 Theorem</u> (Auslander-Reiten formulae): There are isomorphisms, functorial in $X, Y \in A$-mod:

(a) $D\,\underline{\mathrm{Hom}}_A(X,Y) \simeq \mathrm{Ext}^1_A(Y, D\,\mathrm{Tr}X)$

(b) $D\,\mathrm{Ext}^1_A(X,Y) \simeq \overline{\mathrm{Hom}}_A(Y, D\,\mathrm{Tr}X)$

(Note that X and Y change places under both isomorphisms.)

Easy cases: X projective $\Rightarrow \underline{\mathrm{Hom}}_A(X,Y) = 0 \Rightarrow \mathrm{Ext}^1_A(Y, D\,\mathrm{Tr}X) = 0$, which also follows from $D\,\mathrm{Tr}X$ being injective

Y projective $\Rightarrow \underline{\mathrm{Hom}}_A(X,Y) = 0 \Rightarrow \mathrm{Ext}^1_A(Y, D\,\mathrm{Tr}X) = 0$, as we know

What's the point of taking vectorspace duals? On $\mathrm{Hom}_A(X,Y)$ the two endomorphism rings $\mathrm{End}_A(X)$ and $\mathrm{End}_A(Y)$ act on the two sides. D changes the two sides and the isomorphism in (a) lets $\mathrm{End}_A(Y)$ and $\mathrm{End}_A(X)$ act on $\mathrm{Ext}^1_A(Y, D\,\mathrm{Tr}X)$, where $\mathrm{End}_A(Y)$ acts already – and one can check: in the same way.

All these isomorphisms (also in 10.3) can be upgraded to module or bimodule isomorphisms.

Proof of 10.4: Choose P projective mapping onto Y and let

$\delta : 0 \to \underset{\ker}{\underset{\shortparallel}{Q}} \xrightarrow{f} P \xrightarrow{g} Y \to 0$ be exact.

Since P is projective, $Ext^1_A(Y, DTrX) = \delta_*(DTrX)$ by definition of δ_*. Thus we have to identify $D\underline{Hom}(X,Y)$ with $\delta_*(DTrX)$ or, by 1.3, with $D\delta^*(X)$. By definition of δ^*, $\delta^*(X)$ is the quotient of $Hom_A(X,Y)$ modulo the image of $Hom_A(P,Y)$ under g_*. We check that this quotient is exactly $\underline{Hom}_A(X,Y)$: Of course, a map $X \to Y$ in the image of g_* factors through a projective module, P. Conversely, when $h: X \to Y$ factors through P' projective, then

$$\begin{tikzcd} X \arrow[dr] & & Y \\ & P' \arrow[ur, "h'"'] & \end{tikzcd}$$

h' factors through g.

$$\begin{tikzcd} & P' \arrow[dl, dashed, "\exists"'] \arrow[d, "h'"] & \\ P \arrow[r, "g"] & Y \arrow[r] & 0 \end{tikzcd}$$

This proves (a).

(b) can be shown dually. □

These formulae may look exotic, but they turn out to be extremely powerful.

Example: Let X be indecomposable, not projective. Then the identity morphism $id: X \to X$ cannot factor through a projective: Assume it does so, then

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[dr] & X \arrow[r, "id"] \arrow[dr, "t"] & X \\ & P \arrow[u, "s"] \text{ projective} & P \arrow[u, "s"'] \end{tikzcd}$$

Then s is surjective, t injective and $e := t \circ s = e^2$ an idempotent in $End_A(P)$ $\Rightarrow X = Im(e)$ is a direct summand of P $\Rightarrow X$ projective ↯.

$\Rightarrow D\,\underline{Hom}_A(X,Y) \neq 0 \overset{1.4(a)}{\Rightarrow} Ext^1_A(X, DTrX) \neq 0$, so there exists a non-split ses $0 \to DTrX \to ? \to X \to 0$

This is much stronger information than what we knew before (X not projective $\Rightarrow \exists$ non-split $0 \to ? \to ?? \to X \to 0$). This answers Question 1 in the preview.

In the next chapter we will exploit these formulae.