Preview

In 1.6 we have defined the first extension group $\mathrm{Ext}_A^1(Z,X)$ (for A-modules X and Z) to be the set of equivalence classes of extensions of Z by X.

Theorem 2.5 shows that $\mathrm{Ext}_A^1(Z,-)=0 \Leftrightarrow Z$ is projective. In other words, if Z is not projective there exists at least one X such that $\mathrm{Ext}_A^1(Z,X)\neq 0$.

Question: Can we find such an X (in a systematic way) for given Z?

When $\mathrm{Ext}_A^1(Z,X)\neq 0$, there are non-split short exact sequences

$$0\to X\to Y\to Z\to 0 \quad (*)$$

and Y (or its direct summands) may be new module(s). (When X and Z are finite dimensional, $Y=X\oplus Z$ implies that $(*)$ splits, by an exercise in Chapter 4.)

Question: Suppose A has only finitely many indecomposable modules, up to isomorphism. Can we find them all by starting with some X or Z we know (for instance, simple or projective or injective), finding some non-split short exact sequences $(*)$, using the summands of Y as new X or Z and continuing?

Finding all ses $(*)$ may be too much work.

Question: Are there particular ses $(*)$ we should look for?

To get an algorithm, we also need to ask:

Question: How can we decide that a set of indecomposable modules contains, up to isomorphism, all indecomposable modules?

Our methods so far are not strong enough to answer these questions. What one needs is a good new idea or rather a really good and really new idea.

In this chapter and in the next one we will see this idea and how to use it.
This is a core topic of representation theory, called Auslander-Reiten theory.
(Maurice Auslander, 1926-1994, and Idun Reiten, 1942-)
50 years ago, the indecomposable finite dimensional representations were known for very few algebras. From the mid 1970's, AR theory changed the situation completely.

There are now several ways to develop Auslander-Reiten theory and you may have seen one in another course or seminar. We will take an approach starting with Hom and Ext^1, thus applying material of the previous semester. In Chapter 10 we prove crucial formulae. Afterwards we will apply these to representation theory.