§ 1. Short exact sequences

<u>Setup</u>: R is a ring. Let X and Y be (left) R-modules.
A map $X \xrightarrow{\varphi} Y$ is a left R-module <u>homomorphism</u> (or just: morphism)
$:\Leftrightarrow$ $\varphi: X \to Y$ is a homomorphism of abelian groups, $\varphi(x_1+x_2) = \varphi(x_1)+\varphi(x_2)$ $\forall x_1, x_2 \in X$ and $\varphi(0_X) = 0_Y$, and in addition $\varphi(rx) = r\varphi(x)$ $\forall r \in R, x \in X$.
When φ is also bijective, then φ^{-1} is a module homomorphism, too, and φ and φ^{-1} are called <u>isomorphisms</u>.
The morphism φ has a <u>kernel</u>: $\ker(\varphi) := \{x \in X : \varphi(x) = 0\}$, a submodule of X,
and an <u>image</u> $\operatorname{im}(\varphi) = \{\varphi(x) : x \in X\}$, a submodule of Y.

<u>1.1 Definition</u>: Let $n \in \mathbb{N}, n \geq 2$, or $n = \infty$, and $m \in \mathbb{Z}, m \leq 0$, or $m = -\infty$. A sequence $\to M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} M_{i+2} \to \ldots$ $(i \geq m,\ i+2 \leq n)$, where all M_j are R-modules and all φ_j are R-module homomorphisms is called an <u>exact sequence</u> $:\Leftrightarrow$ $\operatorname{im}(\varphi_i) = \ker(\varphi_{i+1})$ whenever both φ_i and φ_{i+1} are defined.
(One may also talk of a sequence being exact at $i+1$ when just $\operatorname{im}(\varphi_i) = \ker(\varphi_{i+1})$ is satisfied.)

Are the following sequences exact?

$$X \xrightarrow{id} X \xrightarrow{id} X$$

$$X \xrightarrow{0} X \xrightarrow{0} X$$

$$0 \xrightarrow{0} X \xrightarrow{id} X \xrightarrow{0} 0$$

$$\ldots \to X \xrightarrow{id} X \xrightarrow{0} X \xrightarrow{id} X \to \ldots$$

Given any morphism $\varphi: X \to Y$ there are exact sequences

$$0 \to \ker(\varphi) \xrightarrow{incl} X \xrightarrow{\varphi} \operatorname{im}(\varphi) \to 0$$

$$0 \to \operatorname{im}(\varphi) \xrightarrow{incl} Y \xrightarrow{quotient} Y/\operatorname{im}(\varphi) \to 0$$

$$0 \to \ker(\varphi) \xrightarrow{incl} X \xrightarrow{\varphi} Y \xrightarrow{quotient} Y/\operatorname{im}(\varphi) \to 0$$

Write these sequences for $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$, $n \in \mathbb{N}_0$.

so, X and Z are determined from the other parts of the sequence

Check: $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ exact $\Rightarrow$ $X \cong \mathrm{Ker}(g)$, $Z \cong Y/\mathrm{Im}(f)$

- $0 \to X \xrightarrow{\varphi} Y$ is exact
  $\Leftrightarrow$ $\mathrm{Ker}(\varphi) = 0$ $\Leftrightarrow$ φ injective
- $X \xrightarrow{\varphi} Y \to 0$ is exact
  $\Leftrightarrow$ $\mathrm{Im}(\varphi) = Y$ $\Leftrightarrow$ φ surjective
- $0 \to X \to 0$ is exact
  $\Leftrightarrow$ $X = 0$
- $0 \to X \xrightarrow{\varphi} Y \to 0$ is exact
  $\Leftrightarrow$ φ is an isomorphism

0 means $\{0\}$, the trivial module

1.2 Definition: An exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is called a short exact sequence. It splits (or is split exact) $:\Leftrightarrow$ in addition, there exists a module homomorphism $h: Z \to Y$ such that $g \circ h = id_Z$.
(Abbreviation: ses = short exact sequence.)

Given X and Z we always can form a split exact sequence

$$0 \to X \xrightarrow{f} X \oplus Z \xrightarrow{g} Z \to 0 \quad \text{where } f: x_0 \mapsto (x_0, 0) \text{ (inclusion)}$$
$$\text{and } g: (x_1, z_1) \longmapsto z_1 \text{ (projection)}$$

This does not tell us anything about X and Z and how they are related. We consider split exact sequences as trivial examples. Our interest is of course in non-trivial ses, and in their middle terms Y that we call extensions of Z by X. Typical question: Given modules X and Z, can we produce new modules (really new ones) as extensions?

What are the middle terms of split exact sequences?

1.3 Lemma: Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence. Then the following statements are equivalent (for short: TFAE, or: ⓔ):

(a) The sequence splits.

(b) $\exists$ module homomorphism $j: Y \to X$ such that $j \circ f = id_X$.

(c) $\exists$ submodule $U \subset Y$ such that $Y = U \oplus \mathrm{Ker}(g)$.

Moreover, when these conditions are satisfied, $Y \simeq X \oplus Z$.

The equivalence of (a) and (b) shows that the definition of split exact has a symmetry – one can split it at the end (finding h) or at the beginning (finding j).

Proof of 1.3: (a) $\Rightarrow$ (b),(c): Assume $h: Z \to Y$ with $g \circ h = id_Z$ exists.

Then $im(h)$ is a submodule of Y and $im(f) \cap im(h) = 0$ since $g \circ f = 0$ and $g \circ h = id_Z$. Let $y_0 \in Y$ and consider $y_0 - h(g(y_0))$. $\Rightarrow g(y_0) = \underbrace{g(h}_{id}(g(y_0)))$ implies $y_0 - h(g(y_0)) \in \ker(g) = im(f)$

$\Rightarrow y_0 \in im(f) + im(h) \Rightarrow Y \cancel{Z} \simeq im(f) \oplus im(h)$ (direct sum).

To get (b) define

$$\begin{tikzcd} j: Y \arrow[r, "(=)", "\simeq"'] \arrow[rrr, bend right, "j"'] & \underset{\underset{\ker(g)}{\parallel}}{im(f)} \oplus im(h) \arrow[r, "\text{projection}"] & im(f) \arrow[r, phantom, "\simeq"] & X \end{tikzcd}$$

To get (c) set $U = im(h)$.

(b) $\Rightarrow$ (a),(c) is similar. (don't believe such claims, but check them independently – this also improves your understanding of (a) $\Rightarrow$ (b),(c))

(c) $\Rightarrow$ (a),(b): Assume $Y = U \oplus \ker(g)$. As $\ker(g) \simeq im(f)$ we can define $j: Y \to X$ by $Y \xrightarrow{proj} \underset{\underset{im(f)}{\parallel}}{\ker(g)} \xrightarrow{\sim} X$ and $h: Y \xrightarrow{projection} U \xrightarrow{\sim} Y$ (since $U \cap \ker(g) = 0$ and $Y = U + \ker(g)$)

Finally, under these conditions, $Y \simeq X \oplus Z$ (isomorphic, not equal):
$Y = im(f) \oplus im(\cancel{g}\, h)$, $im(f) \overset{f}{\simeq} X$ and $im(h) \simeq Z$ (because of $g \circ h = id_Z$). $\square$

So, the middle terms of split exact sequences always are the direct sum of the outer terms, up to isomorphism. But the converse is not true:
$Y \simeq X \oplus Z \not\Rightarrow$ sequence splits. (However, "$\Rightarrow$" works under some assumptions on R and on the modules.)

Exercise: $\overset{R=\mathbb{Z}}{\checkmark}$ $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \mathbb{Z}/4\mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \to 0$

does not split.

Let $M := \bigoplus_{\text{countable}} (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ and

(*) $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\iota,0)} \mathbb{Z}/4\mathbb{Z} \oplus M \xrightarrow{(p,id)} \mathbb{Z}/2\mathbb{Z} \oplus M \to 0.$

Then: (*) is exact.

(*) does not split.

$\underbrace{\mathbb{Z}/4\mathbb{Z} \oplus M}_{Y} \simeq \underbrace{\mathbb{Z}/2\mathbb{Z}}_{X} \oplus \underbrace{\mathbb{Z}/2\mathbb{Z} \oplus M}_{Z}$.

(Note that this example uses $\infty + 1 = \infty$. Guess what assumptions are needed to rule out such counterexamples. The machinery to be built will allow to prove that $Y \simeq X \oplus Z \Rightarrow$ ses splits under suitable assumptions.)

1.3 tells us in particular that all split extensions of Z by X (ie extensions coming from a split ses) are isomorphic to a direct sum, ie trivial. Thus, they all should become equivalent according to the equivalence relation on extensions that we are going to define now.

1.4 Definition: Two extensions $0 \to X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z_1 \to 0$ and $0 \to X_1 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z_1 \to 0$ } some X_1, Z_1

are called equivalent :$\Leftrightarrow$ $\exists$ module isomorphism $\xi : Y_1 \to Y_2$ such that

$\xi \circ f_1 = f_2$ and $g_2 \circ \xi = g_1$.

(It helps to write this condition as saying that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[r, "g_1"] \arrow[d, "\exists \xi"] \arrow[dr, phantom, "\circlearrowleft"] & Z_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_1 \arrow[r, "f_2"] & Y_2 \arrow[r] & Z_1 \arrow[r] & 0
\end{tikzcd}$$

"$\circlearrowleft$" means: "square commutes"

which means each square commutes, that is the compositions of maps going from X_1 to Y_2 coincide (equality) and the same for going from Y_1 to Z_1.)

In 1.4 we could have asked for a module homomorphism ξ to exist, making the diagram commutative (= commute) instead of an isomorphism. This actually would be the same definition: ξ automatically is an isomorphism. We check this now in the diagram:

Suppose
$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, equal] & Y_1 \arrow[r, "g_1"] \arrow[d, "\xi"] & Z_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_1 \arrow[r, "f_2"] & Y_2 \arrow[r, "g_2"] & Z_1 \arrow[r] & 0
\end{tikzcd}$$
commutes for some module homomorphism ξ.

Claim: ξ is bijective, ie an isomorphism.

ξ is injective. Let $y_0 \in Y_1$, $\xi(y_0) = 0 \Rightarrow g_2(\xi(y_0)) = 0 \Rightarrow id_{Z_1}(g_1(y_0)) = 0$

$\Rightarrow g_1(y_0) = 0 \Rightarrow y_0 \in \ker(g_1) = im(f_1) \Rightarrow \exists x_0 \in X_1 : y_0 = f_1(x_0)$

$\Rightarrow \underset{\substack{\parallel \\ 0}}{\xi(y_0)} = \xi(f_1(x_0)) = f_2(x_0) \Rightarrow x_0 \in \ker(f_2) = 0 \Rightarrow x_0 = 0 \Rightarrow y_0 = 0$

ξ is surjective: Let $y_0 \in Y_2$. $g_2(y_0) \in Z_1 = im(g_1) \Rightarrow \exists y_0' \in Y_1 : g_2(y_0) = g_1(y_0')$

$\Rightarrow g_2(y_0) = id(g_1(y_0')) = g_2(\xi(y_0')) \Rightarrow g_2(y_0 - \xi(y_0')) = 0$

$\Rightarrow y_0 - \xi(y_0') \in \ker(g_2) = im(f_2) \Rightarrow \exists x_0 \in X_1 : y_0 - \xi(y_0') = f_2(x_0) = \xi(f_1(x_0))$

$\Rightarrow y_0 = \xi(y_0' + f_1(x_0))$ □

(This method of proof is called diagram chasing. This is a very common technique in homological algebra – the area we are entering now. It looks frightening at first, but once you get used to it, it is often very easy to carry out. You trace an element around the diagram, and in each step you make the only move that allows you to gain some information. Go through the above proof again and try to see how natural the sequence of arguments is.)

Using the term "equivalent" forces us to prove:

1.5 Lemma: Equivalence of extensions is an equivalence relation.

Proof: reflexivity – use $\xi = id$

symmetry – use ξ^{-1}

transitivity – compose the ξ's. □

Do the split exact sequences form an equivalence class? This question has two parts: Are all split exact sequences equivalent? And is an ses equivalent to a split one itself split exact?

Let $\begin{tikzcd} 0 \arrow[r] & X \arrow[r, shift left, "f"] & Y \arrow[r, shift left, "g"] \arrow[l, shift left, "j"] & Z \arrow[r] \arrow[l, shift left, "h"] & 0 \end{tikzcd}$ be split exact. We show it is equivalent

to $0 \to X \xrightarrow{(id,0)} X \oplus Z \xrightarrow{(0,id)} Z \to 0$.

In the proof of 1.3 we have already found an isomorphism $Y \cong X \oplus Z$, coming from the decomposition $Y = \underbrace{\mathrm{im}(f) \oplus \mathrm{im}(h)}_{\text{submodules of } Y}$

$$\rightsquigarrow \quad \begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & Y \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, "{(f,0)}"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{im}(f) \oplus \mathrm{im}(h) \arrow[r, "{(0,g)}"] \arrow[dr, phantom, "\cong"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, "{(id,0)}"] & X \oplus Z \arrow[r, "{(0,id)}"] \arrow[u, "{(f,h)}"] & Z \arrow[r] & 0 \end{tikzcd}$$

equivalent ses (check maps!)

equivalent ses

commutative diagram

Transitivity ~~implies~~ and symmetry imply the sequences are equivalent.

$\Rightarrow$ All split exact sequences are equivalent.

Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be exact and equivalent ~~a~~ to a split exact sequence, hence also to $0 \to X \to X \oplus Z \to Z \to 0$. We show $0 \to X \to Y \to Z \to 0$ split.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & Y \arrow[r, "g"] \arrow[d, "\xi"] \arrow[dr, phantom, "\cong"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, "{(id,0)}"'] & X \oplus Z \arrow[r, "{(0,id)}"'] & Z \arrow[r] & 0 \end{tikzcd}$$

ξ is an isomorphism, $\xi : Y \xrightarrow{\sim} X \oplus Z \Rightarrow Y$ decompose. But this is not enough, we have to define h or j.

Define h: $\begin{tikzcd} Z \arrow[r, "{(0,id)}"'] \arrow[rr, bend left, "h"] & X \oplus Z \arrow[r, "\xi^{-1}"'] & Y \end{tikzcd}$, then $g \circ h : z_0 \mapsto (0, z_0) \mapsto \xi^{-1}(0, z_0) \overset{g}{\mapsto} g(\xi^{-1}(0, z_0))$

and $g(h(z_0)) = (0,id)(\xi(\cancel{z}\, h(z_0))) = (0,id)(\xi(\xi^{-1}(0, z_0))) = z_0$.

(Similarly, $j : Y \to X$ is defined by ?)

When there is a non-split ses, there must be another equivalence class.

Or, more than one.

Let $R=\mathbb{Z}$ and consider the following ses of $\mathbb{Z}$-modules:

$(*)\quad 0\to\mathbb{Z}\xrightarrow{3\cdot}\mathbb{Z}\xrightarrow{1\mapsto\bar{1}}\mathbb{Z}/3\mathbb{Z}\to 0$ non-split

$(**)\quad 0\to\mathbb{Z}\xrightarrow{3\cdot}\mathbb{Z}\xrightarrow{1\mapsto\bar{2}}\mathbb{Z}/3\mathbb{Z}\to 0$ non-split

Claim: $(*)$ and $(**)$ are not equivalent.

Proof: Assume there is an equivalence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "3\cdot"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z} \arrow[r, "1\mapsto\bar{1}"] \arrow[d, "\xi"] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "3\cdot"] & \mathbb{Z} \arrow[r, "1\mapsto\bar{2}"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$\xi:\mathbb{Z}\to\mathbb{Z}$ must be an isomorphism. It is determined by $\xi(1)$, and it can be surjective only for $\xi(1)=1$ or $\xi(1)=-1$ (otherwise the image of ξ is a proper ideal). When $\xi(1)=1$, $\xi=\mathrm{id}$, the right hand square does not commute. When $\xi(1)=-1$, the left hand square does not commute. $\lightning$

This example tells us something about definition 1.4: We could have defined equivalence of extensions by allowing isomorphisms instead of equalities:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & Y_1 \arrow[r] \arrow[d, "\exists\xi"] \arrow[dr, phantom, "\circlearrowright"] & Z_1 \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & Y_2 \arrow[r] & Z_1 \arrow[r] & 0
\end{tikzcd}$$

(but the outer terms still fixed)

For instance,

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "3\cdot"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z} \arrow[r, "1\mapsto\bar{1}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, "\wr", "\substack{\bar{1}\mapsto\bar{2}\\ \bar{2}\mapsto\bar{1}}"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "3\cdot"] & \mathbb{Z} \arrow[r, "1\mapsto\bar{2}"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

would work. So this definition really would be different.

One could, of course, also allow to replace X_1 by $X_2\cong X_1$ and Z_1 by $Z_2\cong Z_1$:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, "\alpha", "\wr"'] & Y_1 \arrow[r, "g_1"] \arrow[d, "\exists\xi", "\wr"'] & Z_1 \arrow[r] \arrow[d, "\wr", "\beta"'] & 0 \\
0 \arrow[r] & X_2 \arrow[r, "f_2"'] & Y_2 \arrow[r, "g_2"'] & Z_2 \arrow[r] & 0
\end{tikzcd}$$

(α and β given isomorphisms)

Again, this would be different from 1.4.

But we can get back to 1.4 by changing the second sequence to:

$$0\to X_1\xrightarrow{f_2\circ\alpha}Y_2\xrightarrow{\beta^{-1}\circ g_2}Z_1\to 0$$

and then ask for equivalence in the sense of 1.4.

We will see later, what this really means.

1.6 Definition: The set of equivalence classes of extensions of Z by X is denoted by $\mathrm{Ext}^1(Z,X)$ and called the <u>first extension group</u>.

This comes with two mysteries: Why "first"? And why "group"? In this ~~sect~~ section we will solve the second one, by finding an abelian group structure on $\mathrm{Ext}^1(Z,X)$.

An abelian group has a zero element – this should be the class of the split extensions. There also must be an addition. How can we add two short exact sequences $0 \to X \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z \to 0$ and ~~$X \to$~~ $0 \to X \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z \to 0$?

$$(+)\quad 0 \to X \oplus X \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}} Y_1 \oplus Y_2 \xrightarrow{\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}} Z \oplus Z \to 0 \text{ is exact.}$$

But it's an extension of $Z \oplus Z$ by $X \oplus X$, not of Z by X. We somehow have to turn $(+)$ into a ses $0 \to X \to ? \to Z \to 0$, which contains information from both given ses.

This will use two general constructions, which we define now. Afterwards we come back to addition in $\mathrm{Ext}^1(Z,X)$.

1.7 Definition: Let R be a ring and X, Y, Z R-modules.

(a) Let $s: X \to Z$ and $t: Y \to Z$ be module homomorphisms.

A <u>pullback</u> (or: <u>fibre product</u>) is a triple (P, p_1, p_2) where P is a module and $p_1: P \to X$ and $p_2: P \to Y$ are morphisms such that $s \circ p_1 = t \circ p_2$ and for all modules A and all morphisms $f_1: A \to X$ and $f_2: A \to Y$ satisfying ~~$f_1 = p_1 \circ g$ and $f_2 =$~~ $s \circ f_1 = t \circ f_2$, there is a unique morphism $g: A \to P$ such that $f_1 = p_1 \circ g$ and $f_2 = p_2 \circ g$.

Diagrammatically:
$$\begin{tikzcd}
\forall A \arrow[dr, dashed, "\exists! g"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\
& X \arrow[r, "s"'] & Z
\end{tikzcd}$$

The square
$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "t"] \\
X \arrow[r, "s"'] & Z
\end{tikzcd}$$
is called a <u>pullback square</u> (the sign $\ulcorner$ marks such a square)

(6) Let $X \xrightarrow{s} Y$ and $X \xrightarrow{t} Z$ be morphisms. A <u>pushout</u> (or: <u>cofibre product</u>) is a triple (Q, q_1, q_2) where Q is a module $q_1: Y \to Q$ and $q_2: Z \to Q$ are morphisms such that $q_1 \circ s = q_2 \circ t$ and for all modules B and morphisms $f_1: Y \to B$ and $f_2: Z \to B$ satisfying $f_1 \circ s = f_2 \circ t$, there is a unique morphism $g: Q \to B$ such that $f_1 = g \circ q_1$ and $f_2 = g \circ q_2$.

In a diagram:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "q_1"] \arrow[ddr, bend left, "f_1"] & \\ Z \arrow[r, "q_2"] \arrow[drr, bend right, "f_2"'] & Q \arrow[dr, dashed, "\exists! g"] & \\ & & \forall B \end{tikzcd}$$

The square $\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "q_1"] \\ Z \arrow[r, "q_2"'] \arrow[ur, phantom, "\lrcorner"] & Q \end{tikzcd}$ is called a <u>pushout square</u>, marked by $\lrcorner$.

(When P and Q don't have names already, one may write PB instead of P, for pullback, and PO instead of Q, for pushout.)

We need to show existence and uniqueness (up to unique isomorphism) of pullbacks and pushouts.

<u>Construction of the pullback</u> for $\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"] & Z \end{tikzcd}$: $X \oplus Y$ is an R-module, too, (direct sum of abelian groups)
with componentwise R-action $r \cdot (x_0, y_0) := (rx_0, ry_0)$.

Set $P := \{(x_0, y_0) \in X \oplus Y : s(x_0) = t(y_0)\} \subset X \oplus Y$.

Check: P is an R-submodule of $X \oplus Y$.

$s(x_1) = t(y_1)$ and $s(x_2) = t(y_2) \Rightarrow s(x_1 + x_2) = s(x_1) + s(x_2) = t(y_1) + t(y_2)$ $\big(= t(y_1 + y_2)\big)$

$r\, s(x_1) = r\, t(y_1) \Rightarrow s(rx_1) = t(ry_1)$

Set $p_1 : P \to X$ the composition $\underbrace{P \overset{incl}{\subset} X \oplus Y \xrightarrow{proj} X}_{p_1}$

and $\underbrace{P \overset{incl}{\subset} X \oplus Y \xrightarrow{proj} Y}_{p_2}$

Check: $\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$ $\qquad \begin{tikzcd} (x_0, y_0) \in P \arrow[r, mapsto] \arrow[d, mapsto] & y_0 \arrow[d, mapsto] \\ x_0 \arrow[r, mapsto] & s(x_0) = t(y_0) \end{tikzcd}$ ✓

Check the "universal property": $\forall A$, $\forall f_1 : A \to X$ $\forall f_2 : A \to Y$ satisfying $s \circ f_1 = t \circ f_2$ $\exists!\, g : A \to P$ such that $f_1 = p_1 \circ g$ and $f_2 = p_2 \circ g$

By assumption $A \xrightarrow{(f_1,f_2)} X \oplus Y \xrightarrow{\binom{s}{-t}} Z$ is the zero map

The kernel of $X \oplus Y \xrightarrow{\binom{s}{-t}} Z$ is, by definition, P

$\Rightarrow$ (f_1, f_2) has image in P and thus factors through the inclusion $P \subset X \oplus Y$

$\Rightarrow$ $\exists\, g: A \to P$ such that

$$\begin{tikzcd} A \arrow[dr, dashed, "g"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$

is commutative.

g is unique: $a \in A$, $g(a) = (x_0, y_0)$

$\Rightarrow p_2(x_0, y_0) = y_0 = f_2(a)$

and $x_0 = f_1(a)$ ✓

<u>Construction of the pushout</u> for

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, bend right] & Q \arrow[dr, "\exists!"] & \\ & & B \end{tikzcd}$$

This diagram is "dual" to the pullback diagram in the sense that all arrows (maps) go into the opposite direction. The construction and the proof for the pushout work indeed by "dualising" (turning around the arrows) the proof in the case of the pullback.

The defining property of the pullback P was that it is the kernel of $\binom{s}{-t}$.

This guaranteed the factorisation ~~of~~ of $A \xrightarrow{(f_1 f_2)} X \oplus Y$ through P.

Now $X \xrightarrow{(s,-t)} Y \oplus Z \xrightarrow{\binom{f_1}{f_2}} B$ is the zero map and we want a factorisation

$$Y \oplus Z \longrightarrow Q \xrightarrow{g} B$$ with $Y \oplus Z \xrightarrow{(f_1, f_2)} B$, once Q is defined.

Therefore, set $Q := (Y \oplus Z) / _{\{s(x) - t(x) : x \in X\}}$, i.e. the quotient of $Y \oplus Z$ modulo the image of $(s, -t)$.

(Such an object is called the cokernel of $(s,-t)$.)

Check: Q is an R-module

Check $\exists\, g: Q \to B$

Check g is unique

Finally, uniqueness of pullback and pushout, up to (unique) isomorphism.

Suppose P and P' both are pullbacks of
(p_1, p_2) (p_1', p_2')

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ X \arrow[r, "s"] & Z \end{tikzcd}$$

Now we use the pullback property in several situations:

$A := P,\ f_1 = p_1,\ f_2 = p_2$

(*)

$$\begin{tikzcd} P = A \arrow[dr, dashed, "\exists ! g"] \arrow[drr, bend left, "f_2 = p_2"] \arrow[ddr, bend right, "p_1 = f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\ & X \arrow[r, "s"] & Z \end{tikzcd}$$

$g = id_P$ works, g unique $\Rightarrow$ only $g = id_P$ works since P is pullback

$A := P',\ f_1 = p_1',\ f_2 = p_2'$

$$\begin{tikzcd} P' \arrow[dr, dashed, "\exists ! g"] \arrow[drr, bend left, "p_2'"] \arrow[ddr, bend right, "p_1'"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\ & X \arrow[r, "s"] & Z \end{tikzcd}$$

since P is pullback: get $P' \xrightarrow{g} P$

$P', (p_1', p_2')$ pullback: $A = P'$

$$\begin{tikzcd} P' \arrow[dr, dashed, "\exists ! g"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & P' \arrow[r, "p_2'"] \arrow[d, "p_1'"] & Y \arrow[d, "f"] \\ & X \arrow[r, "s"] & Z \end{tikzcd}$$

again, this g must be $id_{P'}$

$A = P$:

$$\begin{tikzcd} P \arrow[dr, dashed, "\exists ! g'"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & P' \arrow[r, "p_2'"] \arrow[d, "p_1'"] & Y \arrow[d, "f"] \\ & X \arrow[r, "s"] & Z \end{tikzcd}$$

since P' is ~~a~~ pullback: get $g' : P \to P'$

So far we gained two new maps $P' \xrightarrow{g} P$ and $P \xrightarrow{g'} P'$, both unique. We want these to be mutually inverse isomorphisms. Putting the above diagrams together we get this commutative diagram, which by removing P', p_1' and p_2' gives the commutative diagram

$$\begin{tikzcd} P \arrow[dr, "g'"] \arrow[ddrrr, bend left, "p_2"] \arrow[dddrr, bend right, "p_1"'] & & & \\ & P' \arrow[dr, "g"] \arrow[drr, bend left, "p_2'"] \arrow[ddr, bend right, "p_1'"'] & & \\ & & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\ & & X \arrow[r, "s"] & Z \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[dr, "g \circ g'"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\ & X \arrow[r, "s"] & Z \end{tikzcd}$$

Compare with (*) and conclude $g \circ g' = id_P$.

Similarly, $g' \circ g = id_{P'}$.

Perhaps this uniqueness proof looks complicated to you. Then you should read it again and try to see that the whole proof just relies on the "universal property" guaranteeing the existence of a unique factorisation g in any situation we choose. When P and P' are given, there are four situations each giving a unique g, and the whole proof is just collecting this information. So, this proof is completely formal and never uses that X, Y, Z etc are R-modules. They may as well be sets or topological spaces or apples or pears. Existence of pullbacks, however, depends on the situation we are in.

Prove uniqueness of pushouts.

Pullback and pushout are perfect tools for handling short exact sequences. Let $\xi: 0 \to A \xrightarrow{\alpha} X \xrightarrow{\beta} B \to 0$ be a short exact sequence (so α = inclusion), representing an equivalence class of extensions of B by A, and let $f: C \to B$ be a module morphism. We define a map $f: \mathrm{Ext}^1(B, A) \to \mathrm{Ext}^1(C, A)$ by

↑ deliberately the same letter

sending

$$\begin{array}{rccccccccc} \xi: & 0 & \to & A & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & B & \to & 0 \\ & & & \| & & \uparrow{\scriptstyle \varepsilon}\ {\scriptstyle \llcorner} & & \uparrow{\scriptstyle f} & & \\ \text{to} \quad f(\xi): & 0 & \to & A & \xrightarrow[\mu]{} & P & \xrightarrow[\delta]{} & C & \to & 0 \end{array}$$

where P is the pullback of

$$\begin{array}{ccc} X & \xrightarrow{\beta} & B \\ & & \uparrow{\scriptstyle f} \\ & & C \end{array}$$

Check that $f(\xi)$ is exact (and define μ): Since P is the pullback, the maps $P \xrightarrow{\varepsilon} X$ and $P \xrightarrow{\delta} C$ are defined by $P \subset X \oplus C$ being the kernel of $X \oplus C \xrightarrow{\binom{\beta}{-f}} B$, so

$$P \ni (x_0, c_0) \begin{array}{l} \xmapsto{\varepsilon} x_0 \\ \xmapsto{\delta} c_0 \end{array}$$

First we show that δ is surjective: Let $c_0 \in C$ and we look for a preimage. As in the previous diagram chases we follow the principle: At each step do the unique reasonable thing you can do. So: apply f to get $f(c_0) \in B$. Since β is surjective, $\exists\, x_0 \in X$ with $\beta(x_0) = f(c_0) \Rightarrow \beta(x_0) - f(c_0) = 0$

$\Rightarrow (x_0, c_0) \in P$ and $\delta(x_0, c_0) = c_0$ ✓

Secondly we show that $\ker(\delta) = A$: Let $(x_0, c_0) \in \ker(\delta) \subset P$. I.e. $0 = \delta(x_0, c_0) = c_0$, that is, $\underset{(x_0, c_0)}{\overset{\shortparallel}{(x_0, 0)}} \in P = \ker\binom{\beta}{-f}$, which means $x_0 \in X$ and $\beta(x_0) = 0$.

$\Leftrightarrow x_0 \in A$ (assuming α = inclusion, since X is an extension – otherwise replace A by $\alpha(A)$) ✓

There is of course a dual statement, allowing to change the ~~end~~ starting term:

Given $g: A \to D$ (note the change of direction of the arrow):

$$\begin{tikzcd}
\xi: 0 \arrow[r] & A \arrow[r] \arrow[d, "g"'] & X \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
g(\xi): 0 \arrow[r] & D \arrow[r] & Q \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

where Q is the pushout.

is exact (by the "dual" proof)

Now we can define addition (called Baersum, after Reinhold Baer) in $\mathrm{Ext}^1(B,A)$: Choose representatives of two equivalence classes:

$$\xi: 0 \longrightarrow A \xrightarrow{f_1} X \xrightarrow{g_1} B \to 0$$

$$\eta: 0 \longrightarrow A \xrightarrow{f_2} Y \xrightarrow{g_2} B \to 0$$

Form the direct sum, use a pullback and a pushout

$$\begin{tikzcd}
 & 0 \arrow[r] & A \oplus A \arrow[r, "\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}"] & X \oplus Y \arrow[r, "\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}"] & B \oplus B \arrow[r] & 0 \\
 & 0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\nabla"'] & P \arrow[r] \arrow[u] \arrow[d] & B \arrow[r] \arrow[u, "\Delta"'] \arrow[d, equal] & 0 \\
\xi+\eta: & 0 \arrow[r] & A \arrow[r] & Q \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

and get $\xi + \eta \in \mathrm{Ext}^1(B,A)$.

Here, $\Delta: B \to B \oplus B$ is the diagonal morphism $b \mapsto (b,b)$

and $\nabla: A \oplus A \to A$ is the codiagonal map $(a_1, a_2) \mapsto a_1 + a_2$.

1.8 Theorem: $\mathrm{Ext}^1(B,A)$ is an abelian group.

(Part of) proof: There is a lot of things to be checked:

- Addition is well-defined, i.e. it does not depend on the choice of representative of ξ and η.
- Addition is commutative.
- Addition is associative
- There is a zero element.
- Every ~~element~~ element has an inverse.

We only check some of these assertions (and you may ignore this rather tedious checking); see also 5.2 in Baro's book for more information.

Claim: $\xi: 0 \to B \xrightarrow{(0,id)} A\oplus B \xrightarrow{(id,0)} A \to 0$ split exact is the zero element:

Choose $\eta: 0 \to B \xrightarrow{f} X \xrightarrow{g} A \to 0$ any extension.

Form $\eta+\xi$ (if addition is commutative, checking this case is ~~commute~~ sufficient)

$$\begin{tikzcd}
0 \arrow[r] & B\oplus B \arrow[r] \arrow[d, "\nabla"'] & X\oplus A\oplus B \arrow[r] \arrow[d] & A\oplus A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & Q \arrow[r] & A\oplus A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] & P \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd}$$

apply ∇:

and Δ:

Compute the pushout Q and the maps in the second seq.:

$$\begin{tikzcd}
B\oplus B \arrow[r, "{(f,0,id)}"] \arrow[dr, "\nabla"'] & X\oplus A\oplus B \\
& B
\end{tikzcd}$$

Q is the quotient of $X\oplus A\oplus B \oplus B$ modulo the image of $B\oplus B$ under $(f,0,id)-\nabla$. This quotient is isomorphic to $X\oplus A$ (use that there is $id: B\to B$ and also ∇ is the identity on each summand of $B\oplus B$).

$\Rightarrow$ $0\to B\to Q\to A\oplus A \to 0$ is $0\to B\xrightarrow{f} X\oplus A \xrightarrow{g} A\oplus A \to 0$, $A = A$

and (again because of an identity map $A=A$ and the definition of Δ) the third sequence is $0\to B \xrightarrow{f} P=X \xrightarrow{g} A \to 0$, which is the sequence we started with.

Claim: The additive inverse of $\xi: 0\to B\xrightarrow{f} X \xrightarrow{g} A\to 0$ is $\xi': 0\to B \xrightarrow{f} X \xrightarrow{-g} A \to 0$

To verify this we have to show that $\xi+\xi'$ splits.

$$\begin{tikzcd}
0 \arrow[r] & B\oplus B \arrow[r, "f \oplus f"] \arrow[d, "\nabla"'] & X\oplus X \arrow[r, "g \oplus -g"] \arrow[d] & A\oplus A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & Q \arrow[r] & A\oplus A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] & P \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd}$$

Here, $Q \simeq X\oplus \underbrace{X/f(B)}_{\simeq A}$

Here, $P\simeq B\oplus A$, and the sequence splits. □

How far did we get so far with the problem of describing extensions?
We have defined equivalence of extensions and the set $Ext^1(B,A)$ of extensions (up to equivalence) $0 \to A \to ? \to B \to 0$ of B by A.
We have found algebraic structure on this first extension group: It is an abelian group, and zero is the class of split extensions.
This looks like we are on the right way.
But we have not yet reached our goals:
We don't know yet how to compute $Ext^1(B,A)$ when we are given A and B (and the ring R, of course).
The abelian group structure of $Ext^1(B,A)$ looks terribly complicated. We should look for a better description of the addition, which is as easy (or complicated) as adding for instance two module homomorphisms.

We will reach both goals, but we first have to learn more techniques and find some new connections. Among the techniques are the diagram lemmas; some examples of diagram lemmas, including the most important one for us, are collected in the appendix to this chapter.