Practising universal properties

(1) Let $f: X \to Y$ be an R-module homomorphism.
Show that $\mathrm{Ker}(f)$ and the inclusion $i: \mathrm{Ker}(f) \to X$ satisfy the following property:

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "i"] \arrow[rr, bend left, dashed] & X \arrow[r, "f"] & Y \\
\forall Z \arrow[u, dashed, "\exists! h"] \arrow[ur, "\forall g"'] \arrow[urr, dashed, bend right] & &
\end{tikzcd}$$

$f \circ i = 0$ (this is the meaning of $\xrightarrow{i}\xrightarrow{f}$)
and $\forall Z\ \forall g: Z \to X$ R-module homomorphism with $f \circ g = 0$
$\exists! h\ Z \to \mathrm{Ker}(f)$ R-module homomorphism such that $g = i \circ h$

(In other words: Every g precomposing to zero with f factors uniquely through i.)

Suppose $K \xrightarrow{j} X$ has the same property:

$$\begin{tikzcd}
K \arrow[r, "j"] \arrow[rr, bend left, dashed] & X \arrow[r, "f"] & Y \\
\forall Z \arrow[u, dashed, "\exists! h"] \arrow[ur, "g"'] \arrow[urr, dashed, bend right] & &
\end{tikzcd}$$

Then there is a unique isomorphism $\alpha: K \to \mathrm{Ker}(f)$ making
$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "i"] & X \\
K \arrow[u, dashed, "\alpha"] \arrow[ur, "j"'] &
\end{tikzcd}$$
commute.

(2) Define the cokernel $\mathrm{Cok}(f)$ dually:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, dashed] \arrow[dr, dashed, bend right] & Y \arrow[r, "\pi"] \arrow[d, "\forall g"] & \mathrm{Cok}(f) \arrow[dl, dashed, "\exists! h"] \\
& \forall Z &
\end{tikzcd}$$

Show that $\mathrm{Cok}(f)$ exists and is unique up to unique isomorphism.

Note that $\mathrm{Ker}(f)$ is a pair: $(\mathrm{Ker}(f), i)$, an object with a morphism.
And $\mathrm{Cok}(f)$ is a pair $(\mathrm{Cok}(f), \pi)$.
Thus one can define $\mathrm{Cok}(\mathrm{Ker}(f)) := \mathrm{Cok}(i)$ and $\mathrm{Ker}(\mathrm{Cok}(f)) := \mathrm{Ker}(\pi)$.

(3) Show that $\mathrm{Ker}(\mathrm{Cok}(f)) \cong \mathrm{Cok}(\mathrm{Ker}(f))$ and both are isomorphic to $\mathrm{Im}(f)$ (which in this way can be described abstractly).

Hint:

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\bar{\pi}"] & \mathrm{Cok}(f) \\
& \mathrm{Cok}(i) \arrow[ur, dashed, "\exists"] \arrow[r, dashed, "\exists!"] & \mathrm{Ker}(\bar{\pi}) \arrow[u] &
\end{tikzcd}$$

(4) What is the problem when we try to do the same for groups instead of R-modules?