§1A. Appendix on diagram lemmas

The purpose of this appendix is twofold. The statements collected here will be useful lateron, and the most important result, the Snake Lemma, will provide insight into extension groups. At the same time, proving the results is a good way to practise diagram chasing.
These proofs can easily be found in books or in wikipedia.

1A.1 Lemma (the short five lemma): In the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \to & 0 \\ & & \downarrow g & & \downarrow f & & \downarrow h & & \\ 0 & \to & A' & \xrightarrow{\mu} & B' & \xrightarrow{\delta} & C' & \to & 0 \end{array}$$

assume that g and h are isomorphisms. Then f is an isomorphism, too.

Proof: f is injective: $f(b)=0 \Rightarrow \delta(f(b))=0 \Rightarrow h(\beta(b))=0 \Rightarrow \beta(b)=0$
$\Rightarrow \exists a: b=\alpha(a) \Rightarrow f(\alpha(a))=0 \Rightarrow \mu(g(a))=0$
$\Rightarrow g(a)=0 \Rightarrow a=0 \Rightarrow b=0$ ✓

f is surjective: $b'\in B' \Rightarrow \delta(b')\in C' \Rightarrow \exists c: \delta(b')=h(c)$
$\Rightarrow \exists b: \delta(b')=h(c)=h(\beta(b))=\delta(f(b))$
$\Rightarrow \delta(b'-f(b))=0 \Rightarrow \exists a': b'-f(b)=\mu(a')$
$\Rightarrow \exists a: b'-f(b)=\mu(g(a))=f(\alpha(a))$
$\Rightarrow b'=f(b+\alpha(a))$ ✓ $\square$

compare page 1.5

Note: the existence of f has to be assumed.
For instance,

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/4\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0 \\ & & \| & & & & \| & & \\ 0 & \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0 \end{array}$$

satisfies all other assumptions.

1A.2 Lemma (the five lemma): Given a commutative diagram with exact rows

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow \ell & & \downarrow m & & \downarrow n & & \downarrow p & & \downarrow q \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

Assume m and p isomorphisms, ℓ surjective and q injective. Then n is an isomorphism.

Proof: Practise diagram chasing.

This implies the short five lemma.

<u>1A.3 Theorem</u> (the Snake Lemma): Given a commutative diagram with exact rows

$$(*)\qquad \begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & \end{tikzcd}$$

Then there is an exact sequence

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{\delta} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c)$$

recall: $\operatorname{coker}(a) = A'/\operatorname{im}(a)$

(The map δ is called <u>the connecting homomorphism</u>. Its existence is a crucial and unexpected result.)

To see the snake (and the role of δ), expand $(*)$ into a bigger diagram

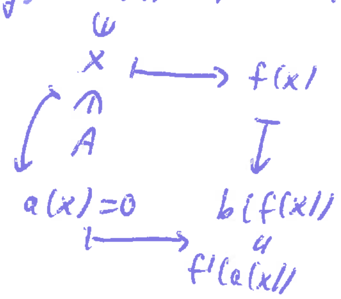
$$\begin{tikzcd} & \ker(a) \arrow[r] \arrow[d, "\mathrm{incl}"'] & \ker(b) \arrow[r] \arrow[d, "\mathrm{incl}"'] & \ker(c) \arrow[dddll, bend left, "\delta"] & \\ & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d, "\mathrm{proj}"'] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\ & \operatorname{coker}(a) \arrow[r] & \operatorname{coker}(b) \arrow[r] & \operatorname{coker}(c) & \end{tikzcd}$$

(with many new morphisms!)

One can say a bit more: When f is injective, then $\ker(a) \to \ker(b)$ is injective, too. When g' is surjective, then so is $\operatorname{coker}(b) \to \operatorname{coker}(c)$.

Proof of 1A.3: The most interesting part is the construction of the new maps, in particular of δ.

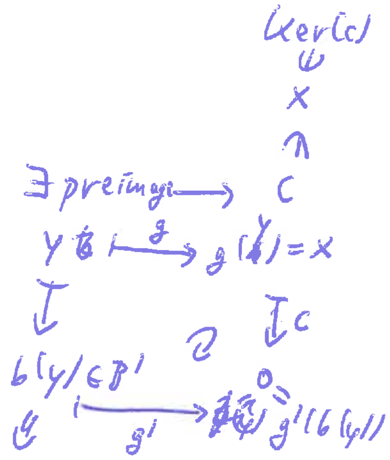
E.g. $\mathrm{Ker}(a) \to \mathrm{Ker}(b)$



$\Rightarrow f(x) \in \mathrm{Ker}(b)$, so the map $\mathrm{Ker}(a) \to \mathrm{Ker}(b)$ is just the restriction of f to $\mathrm{Ker}(a)$..
Similarly, $\mathrm{Ker}(b) \to \mathrm{Ker}(c)$ is the restriction of g.

To define $\mathrm{coker}(a) \to \mathrm{coker}(b)$ let $y \in \mathrm{coker}(a)$, choose a preimage in A', map it to B' by f' and then take the residue class in $\mathrm{coker}(b)$. This has to be shown to be well-defined: Two preimages differ by some $a(x)$ $(x \in A)$. Since $f'(a(x)) = b(f(x))$, the two f'-images coincide modulo $\mathrm{im}(b)$.
Similarly for $\mathrm{coker}(b) \to \mathrm{coker}(c)$.

Now comes the definition of δ. This illustrates nicely the general advice always to make the one reasonable move.



$\exists z \in A'$: $b(y) = f'(z)$

Set $\delta(x) :=$ residue class of z in $\mathrm{coker}(a)$

We need to check that δ is well-defined: Choosing another g-preimage y' of x means $y - y' \in \mathrm{im}(f) = \mathrm{ker}(g)$ and $a(y-y')$ becomes zero in $\mathrm{coker}(a)$.
It remains to see that all the new maps are homomorphisms and to check that the long sequence is exact.

For instance, exactness at $\mathrm{Ker}(c)$: $\delta(x) = 0 \Leftrightarrow z \in \mathrm{im}(a) \Leftrightarrow b(y) \in \mathrm{im}(b \circ f)$
When $x = g(\underset{\mathrm{Ker}(b)}{u})$, then we can choose as preimage, i.e. $y = u$ and then $b(y) = 0$
Conversely, when $b(y) \in \mathrm{im}(b \circ f)$ then $y \in \underbrace{\mathrm{im}(f)}_{\mathrm{Ker}(g)} + \mathrm{Ker}(b)$
$\Rightarrow g(y) = x \in g(\mathrm{Ker}(b))$ ✓ $\square$

The short five lemma is a special case of the snake lemma, with a and c isomorphisms, f injective and g' surjective. Then $\ker(a) = 0$, $\ker(c) = 0$, $\operatorname{coker}(a) = 0$ and $\operatorname{coker}(c) = 0$, by assumption, and the exact sequence in 1A.3 becomes $0 \to \ker(b) \to 0 \to 0 \to \operatorname{coker}(b) \to 0$. Therefore, $\ker(b) = 0$ and $\operatorname{coker}(b) = 0$ and b must be an isomorphism.