A New Characterization of Algebras of Colocal Type

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Colocal type

- Let $\mathcal{A}$ be an abelian category where every object is of finite length.
- An object $X \in \mathcal{A}$ is called *colocal* if its socle is simple.
- Equivalently, it is colocal if every non-zero subobject of X is indecomposable.

Definition

- A category $\mathcal{A}$ is *of colocal type* if every indecomposable object in $\mathcal{A}$ is colocal.
- An Artin algebra A is *of colocal type* if $\mathcal{A} = \operatorname{mod} A$ is of colocal type.

History

- Algebras of colocal type have been studied repeatedly
- A first (complicated) characterization dates back to Tachikawa in 1959
- Two gaps in the proof were filled by Sumioka in 1984

A characterization

Definition

For all simple objects $S, T \in \mathcal{A}$ let

$$d^1_S(S, T) := \dim_{\mathrm{End}(S)^{op}} \mathrm{Ext}^1(S, T)$$

and

$$d^1_T(S, T) := \dim_{\mathrm{End}(T)} \mathrm{Ext}^1(S, T).$$

Definition

Given simple object S, S' with $\mathrm{Ext}^1(S, S') \neq 0$, let

$$\mathcal{T}_{S,S'} := \left\{ T \text{ simple and } \mathrm{Ext}^1(T, S) \neq 0 \mid \exists Z : \begin{cases} l(Z) = 3 \\ \operatorname{soc} Z = S' \\ \operatorname{top} Z = T \end{cases} \right\}$$

Theorem

$\mathcal{A}$ is of colocal type if and only if the following conditions hold for all simple objects $S \in \mathcal{A}$:

(C1)

$$\sum_{T \text{ simple}} d_T^1(S,T) \leq 1$$

(C2)

$$\sum_{T \text{ simple}} d_T^1(T,S) \leq 2$$

(C3) *If there is a simple object S' with $\operatorname{Ext}^1(S,S') \neq 0$, then*

$$\sum_{T \in \mathcal{T}_{S,S'}} d_T^1(T,S) \leq 1.$$

Suppose that $\mathcal{A}$ fulfils (C1).

$$\mathcal{T}_{S,S'} = \mathcal{T}'_{S,S'} := \left\{ T \text{ simple and } \operatorname{Ext}^1(T,S) \neq 0 \mid \exists Z : \begin{cases} l(Z) \geq 3 \\ \operatorname{soc} Z = S' \\ \operatorname{top} Z = T \end{cases} \right\}.$$

If $S \ncong S'$, then $\mathcal{T}_{S,S'} = \mathcal{T}''_{S,S'}$ with

$$\mathcal{T}''_{S,S'} := \left\{ T \text{ simple and } \operatorname{Ext}^1(T,S) \neq 0 \mid \exists Z : \operatorname{soc} Z = S', \operatorname{top} Z = T \right\}.$$

If no cycle in the Ext-quiver has S' as a vertex, then

$$\mathcal{T}_{S,S'} = \mathcal{T}'''_{S,S'} := \left\{ T \text{ simple} \mid \operatorname{Ext}^1(T,S) \neq 0 = \operatorname{Ext}^2(T,S') \right\}.$$

Algebras over algebraically closed fields

Let $A := kQ/I$ for a quiver Q and an admissible ideal I.

Corollary

A is of colocal type if and only if

- *No vertex in Q is starting point of more than one arrow.*
- *No vertex in Q is end point of more than two arrows.*
- *Given an arrow β, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$.*

Proposition

A is of colocal type if and only if

- *No vertex in Q is starting point of more than one arrow.*
- *A is a string algebra.*

The condition (C1)

- (C1) $\iff$ every module with simple top is uniserial
- If

$$\mathcal{A} = \operatorname{mod} A$$

for a finitely generated algebra A, then

$$\text{(C1)} \iff \text{every indecomposable projective } A\text{-module is uniserial}$$

- If

$$\sum_{T \text{ simple}} d_T^1(S, T) = \sum_{T \text{ simple}} \dim_{\operatorname{End}(T)} \operatorname{Ext}^1(S, T) = n,$$

then there is a projective module with socle S^n.

Idea of Proof

- $\mathcal{A}$ is of colocal type $\Longrightarrow$ (C2) - (C3): diagram chasing

If (C1) - (C3) is fulfilled and $X \in \mathcal{A}$ is indecomposable, but not uniserial, then

- X has a simple socle S with

$$\sum_{T \text{ simple}} d_T^1(T, S) = 2$$

- $X/S \cong U_1 \oplus U_2$, where U_1 and U_2 are uniserial.

Idea of Proof

- We show this first for colocal X
- If there is a monomorphism

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} : S_1 \oplus S_2 \rightarrowtail X,$$

  then

$$X \rightarrowtail \operatorname{Coker} f_1 \oplus \operatorname{Coker} f_2$$

- We can assume that all indecomposable direct summands of $\operatorname{Coker} f_1 \oplus \operatorname{Coker} f_2$ are colocal.
- We show that X is decomposable using the form of those direct summands

Submodule closed subcategories

Definition

A full, additive subcategory $\mathcal{C}$ of an abelian category is called *subobject closed*, if

$$X \in \mathcal{C}, Y \subset X \Rightarrow Y \in \mathcal{C}$$

The subobject closed subcategories of $\mathcal{A}$ form a lattice $\mathbf{S}(\mathcal{A})$:

- meet: $\text{obj}\ (\mathcal{C} \wedge \mathcal{D}) = \text{obj}\ \mathcal{C} \cap \text{obj}\ \mathcal{D}$
- join: $\text{obj}\ (\mathcal{C} \vee \mathcal{D}) = \{X \in \mathcal{A} \mid \exists C \in \mathcal{C}, D \in \mathcal{D} : X \subset C \oplus D\}$

Example: $A = kQ$, $Q = 1 \leftarrow 2 \rightarrow 3$

- $\{100, 001, 111\}$ is submodule closed
- $\{100, 110, 010\}$ is submodule closed
- $\{100, 011, 110, 011\}$ is not submodule closed, since

$$111 \subset 110 \oplus 011$$

Definition

A lattice L is called *distributive* if

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

for all $a, b, c \in L$.

Definition

For objects $X, Y \in \mathcal{A}$

$$X \in \operatorname{sub} Y \iff \text{ there is a monomorphism } X \rightarrowtail Y^{\,n} \text{ for some } n \in \mathbb{N}$$

Proposition

The following statements are equivalent:

- *The lattice* $\mathrm{S}(\mathcal{A})$ *is distributive.*
- *If* $X \in \mathcal{A}$ *is indecomposable and there are objects* $Y_1, Y_2 \in \mathcal{A}$, *so that*

  $$X \in \operatorname{sub} Y_1 \vee \operatorname{sub} Y_2$$

  then

  $$X \in \operatorname{sub} Y_i$$

  for some $1 \leq i \leq 2$.

Proposition

If $\mathcal{A}$ is of colocal type, then $\mathsf{S}(\mathcal{A})$ *is distributive.*

Proof.

- Let X be indecomposable with a monomorphism

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \rightarrowtail \operatorname{Im}(f_1) \oplus \operatorname{Im}(f_2).$$

- Then there is a monomorphism $\operatorname{Ker} f_1 \oplus \operatorname{Ker} f_2 \rightarrowtail X$
- So one of f_1, f_2 is a monomorphism.

□

Partitions

- The partitions of natural numbers form a lattice, called *Young's lattice*
- Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$ and $\lambda' := (\lambda'_1, \lambda'_2, \lambda'_3, \dots, \lambda'_{n'})$ be partitions.
  Suppose that $n \leq n'$ and set $\lambda_i := 0$ for $i > n$. Then

  $$\lambda' \wedge \lambda = \lambda \wedge \lambda' = (\min(\lambda_1, \lambda'_1), \dots, \min(\lambda_n, \lambda'_n)))$$

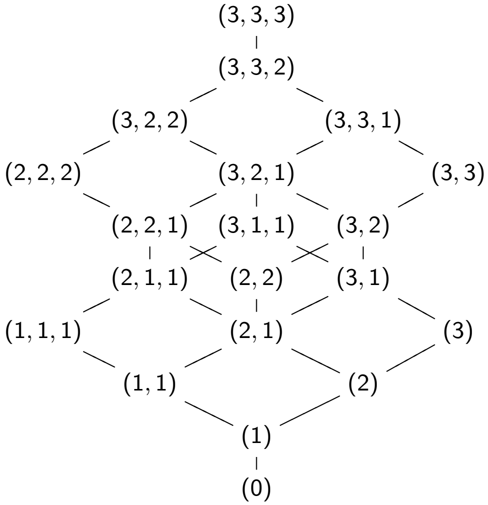
  and

  $$\lambda' \vee \lambda = \lambda \vee \lambda' = (\max(\lambda_1, \lambda'_1), \dots, \max(\lambda_{n'}, \lambda'_{n'}))).$$

Definition

$Y^{m,n}$ is the lattice of all partitions with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{m'})$ where $m' \leq m$ and $\lambda_i \leq n$ for all $1 \leq i \leq m'$.

Example: $Y^{3,3}$



The Lattice $S(\text{mod}\, A)$

- Let $A \cong kQ/I$ and A be of colocal type.
- For every vertex m of Q, there are at most two maximal paths without relations that end in m.
- If two paths end in m, let k_m and l_m be their lengths.
- If one path ends in m, let k_m be its length and $l_m = 0$.

Theorem

Suppose that $A \cong kQ/I$. If A is of colocal type, then

$$\mathrm{S}(\text{mod } A) \cong \prod_{m \text{ vertex of } Q} Y^{k_m+1, l_m+1}.$$