Weyl Module Objects that are of fundamental importance for the representation theory of reductive algebraic groups G (cf. **Representations of a group; Reductive group; Algebraic group**). Considering such groups as group schemes (cf. **Group scheme**), that is as family of groups $G(K)$, where K varies over some class of commutative rings, Weyl modules are universally defined and possess a standard basis which is independent of the choice of K. Moreover, for algebraically closed fields K of characteristic 0 they constitute a complete set of nonisomorphic irreducible rational representations of $G(K)$. This makes possible a "modular theory" for the rational representations of these groups analogous to Brauer's modular representation theory of finite groups. However, whereas for finite groups the reduction modulo a prime p happens with respect to the field of coefficients of the representations, in the algebraic group case this reduction is carried out with respect to the field of definition of the group.

Below the example of general linear groups will be discussed in more detail to illuminate this reduction process. For these R. Carter and G. Lusztig used the term "Weyl module" the first time in their fundamental paper [5], where they discussed polynomial representations of general linear groups and indicated, how their methods generalize to arbitrary reductive groups. There these modules were constructed in the "same" way as in H. Weyl's book [19].

General linear groups. Let $n \in \mathbb{N}$ and let $G = GL_n(K)$ for some field K (cf. **General linear group**). Let $E = K^n$ be the natural module of G. For a nonnegative integer r the group G acts on $E^{\otimes r}$ diagonally, and this action centralizes the natural place permutation action of the symmetric group $\mathfrak{S}_r$ on $E^{\otimes r}$. The image of KG in $\mathrm{End}_K(E^{otimesr}$ is the Schur algebra $S(n,r)$. This is a finite dimensional quasi hereditary K-algebra and the indecomposable representations of $S(n,r)$ for $0 \leq r \in \mathbb{Z}$ are precisely the indecomposable polynomial representations of G. For fields K of characteristic 0 the polynomial representations of G have been investigated by Schur in his famous dissertation [14]. He rederived all these results in a paper [15] of 1927 in terms of the module $E^{\otimes r}$. Among other things he showed $E^{\otimes r}$ is a completely reducible G-module, indeed the category of polynomial representations of G is semisimple. Weyl modules are defined to be the irreducible constituents of tensor space $E^{\otimes r}$ where r runs through $\mathbb{N} \cup \{0\}$. However Weyl modules are defined for arbitrary characteristic $p \geq 0$ of the field K. Those occuring for $p = 0$ as irreducible constituents of the tensor space $E^{\otimes r}$ are parametrized by partitions λ of r into n parts. Thus $\lambda = (\lambda_1, \dots, \lambda_n)$ is a decreasing sequence of nonnegative integers whose sum is r. If we drop the requirement "decreasing" in the definition above, we call λ a composition of r into n parts. The union $\Lambda(n)$ of the sets $\Lambda(n,r)$ of compositions of r into n parts, where $0 \leq r \in \mathbb{Z}$, is the set of weights of G. The partitions in $\Lambda(n,r)$ are also called dominant weights and are denoted by $\Lambda^+(n,r)$. We depict the composition λ into n parts by its associated Young diagram consisting of crosses in the plane in n rows, where the ith row contains λ_i crosses without gap and the first crosses in the rows are the crosses in the first column. We denote this Young diagram by $[\lambda]$.

Let λ be a composition of r. A λ-tableau t is a Young diagram of shape λ, where the crosses are replaced by the numbers $1, 2, \dots, r$ in some order (cf. **Young tableau**). The λ-tableau, where these numbers are inserted in order along the rows downward is called the initial λ-tableau and is denoted by t^λ.

A λ-tableau is row standard, if the numbers increase along the rows and column standard, if they increase down the columns. A row and column standard tableau is called standard. Obviously the initial tableau is always standard. The symmetric group $\mathfrak{S}_r$ acts on the set of λ-tableaux where λ is a composition of r. If t is a λ-tableau the set $R(t)$ of elements of $\mathfrak{S}_r$ permuting the entries in each row is a subgroup of $\mathfrak{S}_r$, the row stabilizer of t. Similarly we define the column stabilizer $C(t)$ of t. In particular for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda(n,r)$ the row stabilizer $R(t^\lambda)$ of the initial λ-tableau is the standard Young subgroup Y_λ of $\mathfrak{S}_r$, which is the symmetric group

$$\mathfrak{S}_{\{1,2,\dots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \times \dots \times \mathfrak{S}_{\{\lambda_1+\cdots\lambda_{n-1}+1,\dots,r\}}$$

considered as subgroup of $\mathfrak{S}_r$ in the natural way.

We are now in the position to define Weyl modules for $GL_n(K)$ for arbitrary fields K: For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n,r)$ we define the element e_λ of tensor space $E^{\otimes r}$ to be

$$e_\lambda = \underbrace{e_1 \otimes e_1 \otimes \cdots \otimes e_1}_{\lambda_1-\text{factors}} \otimes e_2 \otimes \cdots \otimes \underbrace{e_n \otimes e_n \otimes \cdots \otimes e_n}_{\lambda_n-\text{factors}},$$

the element y_λ of the group algebra $K\mathfrak{S}_r$ to be the alternating sum

$$y_\lambda = \sum_{\pi \in C(t^\lambda)} sg(\pi)\pi,$$

where $sg(\pi)$ denotes the signum of π, and finally

$$z_\lambda = e_\lambda y_\lambda \in E^{\otimes r}.$$

For $\lambda \in \Lambda^+(n,r)$ the Weyl module to highest weight λ is the left G-submodule of tensor space $E^{\otimes r}$ generated by z_λ and is denoted by $\Delta(\lambda)$. Thus

$$\Delta(\lambda) = KGL_n(K)z_\lambda,$$

provided K is infinite.

The vector z_λ is an eigenvector of the the torus T of diagonal matrices in G. The eigenvalue is a linear character of T given as

$$\mathrm{diag}(t_1, t_2, \dots, t_n) \mapsto t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n} \in K,$$

which again is denoted by λ (cf. also **Character of a group**). Thus the weights $\Lambda(n)$ of $GL_n(K)$ are in bijection with the linear characters of T. The module $\Delta(\lambda)$ decomposes into direct sum of eigenspaces of the action of T, called weight spaces and for $\lambda \in \Lambda^+(n,r)$ all occurring weights belong to $\Lambda(n,r)$. Let $\Delta(\lambda)^\mu$ denote the eigenspace of $\Delta(\lambda)$ to weight μ, and define the formal character of $\Delta(\lambda)$ to be the formal sum

$$\chi_\lambda = \sum_{\mu \in \Lambda(n)} \dim_K(\Delta(\lambda)^\mu) e_\mu,$$

where e_μ denotes a formal basis element of the integral group ring $\mathbb{Z}\Lambda(n)$ corresponding to μ. The formal character χ_λ is known by the formulas of Weyl, Kostant and Freudenthal (see also **Character formula**). This will be explained later in detail in the more general context of reductive groups. It turns out that λ is the highest weight with respect to the lexicographic order on $\Lambda(n)$ occurring with nonzero multiplicity in $\Delta(\lambda)$ and $\Delta(\lambda)$ and it occurs with multiplicity one. Moreover $\Delta(\lambda)$ is universal with this property, that is every polynomial G-module which is generated by an eigenvector to weight λ such that the corresponding eigenspace is one dimensional and all occurring weights come later in the lexicographic order is an epimorphic image of $\Delta(\lambda)$.

It is obvious that no proper submodule of $\Delta(\lambda)$ can contain the highest weight vector z_λ, hence the sum of all proper submodules is the unique maximal submodule of $\Delta(\lambda)$. The corresponding irreducible factor module is denoted by $L(\lambda)$. It turns out that the set of all $L(\lambda)$, where λ runs through the set $\Lambda^+(n)$ of dominant weights of G, is a complete set of non isomorphic irreducible polynomial G-modules. Tensoring by arbitrary negative powers of the one dimensional polynomial representation of G which takes every matrix to its determinant we obtain a full clasification of all irreducible rational representations of general linear groups.

The space $E^{\otimes r}$ admits a bilinear form $<\ ,\ >$ which is G-invariant and contravariant in the sense that

$$< gx\,,\, y >=< x\,,\, g^t y > \qquad \forall g \in G,$$

where g^t denotes the transpose of g. The space $\Delta^{\lambda\perp}$ orthogonal to the kernel of the natural projection $E^{\otimes r} \to \Delta(\lambda)$ with respect to this form is again a G-module denoted by $\nabla(\lambda)$ and has properties dual to those of Weyl modules. So it has the same formal character and its socle is simple and is isomorphic to $L(\lambda)$.

There is another interpretation of the ∇-modules namely as induced modules from linear representations of a **Borel subgroup** B of G: We may take B to be the set of all upper triangular matrices in G. Then every linear representation of B is a linear character λ of T considered as representation of B, where the unipotent radical U of $B = TU$ acts trivially. The restriction functor to algebraic subgroups H of an algebraic group G has an right adjoint, given by taking for a rational H-module M the set of H-equivariant morphisms from G to M. It is called induction and denoted by Ind_H^G. (The restriction functor has also a left adjoint given by tensoring up representations. But whereas for finite groups the left and right adjoint of the restriction functor are isomorphic, this is not true in the context of algebraic groups, in fact the tensored module is in general not even rational). The G-module ∇^λ is the the induced module $\mathrm{Ind}_B^G(\lambda)$.

For further reference and results on the special case of general linear groups in particular for explicit formulas for bases of Weyl modules and induced modules in terms of bideterminants see J.A. Green's fundamental monograph [8].

Reductive groups.All this generalizes to arbitrary reductive groups. For simplicity we assume that G is simple algebraic group of universal type over some algebraically closed field K. The general case can be derived from this one. For example, the special linear groups $SL_n(K)$ are of this form. The representations of $GL_n(K)$ can be derived from those of $SL_n(K)$ by extending the action of diagonal matriceslocalizing, and those of $PSL_n(K)$ by considering only a certain sublattice of the weight lattice of $SL_n(K)$.

Associated with G we have again a weight lattice Λ which may be considered as group of linear characters of a maximal torus T of G. This is a $\mathbb{Z}$-lattice of rank n, the Lie rank of G, with basis consisting of the so called fundamental weights. There is a partial order on Λ setting $\lambda \leq \mu$ if $\mu - \lambda$ is a linear combination of the fundamental weights with nonnegative coefficients. The linear combinations the fundamental weights, where all coefficients are nonnegative, are called dominant weights. The set of dominant weights is denoted by Λ^+. There are several ways how to define Weyl modules $\Delta(\lambda)$ for dominant weights λ. The two most important of those are briefly described below.

First, one may extend λ to a linear representation of a Borel subgroup $B = TU$, where U is the unipotent radical of B and is in the kernel of λ. Inducing this B-module to G produces $\nabla(\lambda)$. Similar to $GL_n(K)$ the group G admits the notion of a contravariant dual module: Given any rational left module V of G the natural right G-module structure on the dual space $V^* = \mathrm{Hom}_K(V, K)$ can be turned into a left module structure using any antiautomorphism of KG. Usually one considers the map induced by taking inverses in G. For contravariant duality we use a certain different antiautomorphism of G (which given by transposing matrices in the special case of $G = GL_n(K)$). The Weyl module $\Delta(\lambda)$ is then defined to be the contravariant dual module $\nabla(\lambda)^*$. (One can use the ordinary dual as well, but then one has to carry out a shift in the highest weight). For non-dominant weights λ the induced module $\mathrm{Ind}_B^G(\lambda)$ is the zero module. Indeed, since G is assumed to be simple of universal type, one can use this property to define dominant weights.

The second approach for setting up Weyl modules involves the complex simple Lie algebra $\mathfrak{g}$ associated with G. So for example for $G = SL_n(K)$ the associated complex Lie algebra $\mathfrak{g}$ is the set $\mathfrak{sl}_n(\mathbb{C})$ of complex $n \times n$ matrices of trace 0 where Lie bracket is given as usual by commutators. We choose a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, e.g. the set of diagonal matrices in our example $\mathfrak{sl}_n(\mathbb{C})$. The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ determines a root system Φ with basis Σ and corresponding to the choice of Σ a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. We have the root space decomposition

$$\mathfrak{g} = \sum{}^{\oplus}_{\alpha\in\Phi^-} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \sum{}^{\oplus}_{\gamma\in\Phi^+} \mathfrak{g}_\gamma$$

of $\mathfrak{g}$. Thus Φ is a subset of $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ consisting of the nonzero eigenvalues of $\mathfrak{h}$ on $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the eigenspace to eigenvalue $\alpha \in \Phi$. All these eigenspaces are one dimensional and there is a special basis $\mathcal{B}$, called Chevalley basis, of $\mathfrak{g}$, which is not only compatible with this decomposition, but also has particularly nice structural constants:

$$\mathcal{B} = \{e_{\pm\alpha}, h_\beta \mid \alpha \in \Phi^+, \beta \in \Sigma\}.$$

Thus $\ell = |\Sigma|$ is the dimension of $\mathfrak{h}$. For $\mathfrak{sl}_n(\mathbb{C})$ one can choose the root subspaces $\mathfrak{g}_\alpha$ to consist of scalar multiples of matrix units e_{ij}, for $i \neq j$, (positive for $i < j$ and negative for $i > j$). The elements h_β of the Chevalley basis $\mathcal{B}$ of $\mathfrak{sl}_n(\mathbb{C})$ are the matrices $e_{ii} - e_{i+1i+1}$ for $1 \leq i \leq n-1$.

There is a duality between the euclidian space $\mathcal{E}^\ell$ containing the root system Φ (induced by the killing form) and the real space generated by the Chevalley basis of $\mathfrak{h}$. The elements dual to roots are called coroots, and the linear functions on $\mathfrak{h}$ which map the coroots to integers are called weights. The set of weights is a free $\mathbb{Z}$-lattice $\Lambda \supseteq \Phi$ of rank $|\Sigma|$. A weight whose values on positive cooroots are nonnegative is called dominant. The basis of the weight lattice dual to the simple coroots is the set of fundamental weights.

The toral subalgebra $\mathfrak{h}$ acts diagonally on every finite dimensional $\mathfrak{g}$-module M, hence it decomposes into the eigenspaces, called weight spaces M^λ, $\lambda \in \Lambda$. A module which is generated as a $\mathfrak{g}$-module by a weight vector v^+ of weight λ such that all further occurring weights are later in the partial order on Λ is called a highest weight modules of highest weight λ. The generator v^+ is called a maximal vector. There are universal highest weight modules, called Verma modules (cf. **Representation of a Lie algebra**). They play a role similar to Weyl modules for G. Thus for instance, the Verma module $V(\lambda)$ to highest weight λ is induced from the linear representation of the borel subalgebra $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ setting $\mathfrak{n}^+ = \sum^{\oplus}_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, which arises by extending the linear character λ of $\mathfrak{h}$ to $\mathfrak{b}^+$ with trivial $\mathfrak{n}^+$-action. Morover, $V(\lambda)$ admits a contravariant $\mathfrak{g}$-invariant bilinear form and hence has a unique maximal submodule. The corresponding irreducible factor module $W(\lambda)$ is a highest weight module with highest weight λ and the set of all modules $W(\lambda)$, where λ runs through the set Λ, is a complete set of irreducible objects in the category $\mathcal{O}$ associated with $\mathfrak{g}$. Those with dominant highest weight are precisely the finite dimensional ones. The category $\mathcal{O}$ consists of all $\mathfrak{g}$-modules which allow a weight space decomposition with finite dimensional weight spaces and have locally nilpotent e_α-action for all $\alpha \in \Phi$. The weight space decomposition of modules V in $\mathcal{O}$ can be encoded in the formal character χ_V. For dominant weights λ the formal characters and consequently the dimension of the finite dimensional irreducible modules $W(\lambda)$ are known by formulas of Freudenthal, Kostant and Weyl (cf. also **Character formula**).

Let V be a finite dimensional $\mathfrak{g}$-module. Since $\mathfrak{g}$ is simple this module affords a **faithful representation** of $\mathfrak{g}$. The group $\Lambda(V)$ generated by the weights occuring in V lies between the root lattice and the weight lattice Λ. For $\alpha \in \Phi$ the element e_α of the Chevalley basis $\mathcal{B}$ of $\mathfrak{g}$ acts as nilpotent endomorphism on V. Thus we can exponentiate its representing matrix, which we denote as well by e_α. We also want to carry out a base change simultaneously to obtain a transition from $\mathbb{C}$ to an arbitrary field K. For this we consider Kostant's $\mathbb{Z}$-form $\mathcal{U}_\mathbb{Z}$ in the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ (cf. also **Universal enveloping algebra**). This is the $\mathbb{Z}$-subalgebra of $\mathfrak{U}(\mathfrak{g})$ generated by elements of the form $\frac{e_\alpha^i}{i!}$ for $0 \le i \in \mathbb{Z}$ and $\alpha \in \Phi$ and elements of the form $\binom{h}{i} = \frac{h(h-1)\cdots(h-i+1)}{i!}$ for $h = h_\beta \in \mathfrak{h}$, where $\beta \in \Sigma$. The K-algebra $\mathcal{U}_K = K \otimes_\mathbb{Z} \mathcal{U}_\mathbb{Z}$ is called the hyper- or divided power algebra of $\mathfrak{g}$. We choose a $\mathcal{U}_\mathbb{Z}$-invariant $\mathbb{Z}$-lattice M in V. Then for $t \in K$ and $\alpha \in \Phi$ the operator $x_\alpha(t) = \sum_{i=0}^\infty t^i \otimes \frac{e_\alpha^i}{i!}$ is a well defined automorphism of $M_K = K \otimes_\mathbb{Z} M$. The group generated by $\{x_\alpha(t) \mid t \in K, \alpha \in \Phi\}$ is the **Chevalley group** $G_K(V)$. If $\Lambda(V)$ is the root lattice, this group is of adjoint, if $\Lambda(V) = \Lambda$, then it is of universal type and indeed $G_K(V) = G$ is the group which we started with. Any other Chevalley group over K associated with $\mathfrak{g}$ is an epimorphic image of G. In particular, even if $\Lambda(V) \neq \Lambda$, the K-space M_K is a G-module.

We apply this to the irreducible $\mathfrak{g}$-module $W(\lambda)$ for $\lambda \in \Lambda^+$. Let v^+ be a maximal vector in $W(\lambda)$. Then $M = \mathcal{U}_\mathbb{Z} v^+$ is an $\mathcal{U}_\mathbb{Z}$-invariant lattice in $W(\lambda)$ and the G-module M_K is the Weyl module $\Delta(\lambda)$. The weights of G and $\mathfrak{g}$ coincide, hence Weyl's character formula (and the formula of Freudenthal) describe the formal character of $\Delta(\lambda)$. The lattice M is minimal among all $\mathcal{U}_\mathbb{Z}$-invariant lattices in $W(\lambda)$ with the property that it intersects the weight space $W(\lambda)^\lambda$ of λ precisely in $\mathbb{Z}v^+$. This set also has an upper bound $\hat{M}$, and $\nabla(\lambda) = \hat{M}_K$ as G-module.

Character formulas. If K is algebraically closed of characteristic 0 the Weyl modules are irreducible, but this is not true in general for fields K of positive characteristic. However, as in the special case of general linear groups, $\Delta(\lambda)$ has a unique maximal submodule and the set of factor modules $L(\lambda)$ is a complete set of nonisomorphic irreducible rational KG-modules.

One of the most outstanding open problems concerning Weyl modules is to determine the composition factors of those, or in the language of Brauer theory to determine the decompostion matrix of G. Thus let $d_{\lambda\mu}$ be the multiplicity of $L(\mu)$ as composition factor of $\Delta(\lambda)$. From Weyl's character formula one sees immediately that $d_{\lambda\lambda} = 1$ and that $d_{\lambda\mu} \neq 0$ implies that the partition μ comes later in the partial order on weights than λ. Ordering Λ^+ lexicographically the decomposition matrix becomes lower unitriangular. Since the formal characters of Weyl modules are known, the problem of computing the $d_{\lambda\mu}$ is equivalent to the problem of finding the formal characters of the irreducible G-modules $L(\lambda)$.

There is a similar problem in the representation theory of the Lie algebra $\mathfrak{g}$ associated with G, namely the computation of the composition factors of Verma modules or equivalently the computation of the formal character of $W(\lambda)$ for all $\lambda \in \Lambda$ not only the dominant weights λ. D. Kazhdan and G. Lusztig conjectured such a character formula in 1979 in [9], which was proven shortly after in [3] and independently in [4]. The combinatorics in this formula are given by Kazhdan-Lusztig polynomials which are based on properties of the Hecke algebra, a deformation of the Weyl group of G.

The Lusztig conjecture predicts similarly a character formula for Weyl modules in terms of certain Kazhdan-Lusztig polynomials, however under additional assumptions on the characteristic p of K as well as on the highest weights. If confirmed one can derive the decomposition numbers of the other Weyl modules too. In [1] H.H. Andersen, J.C Jantzen and W. Soergel proved the Lusztig conjecture for large p without determining a concrete bound for p. Here is a brief description of the things involved in the proof:

The notion of a **category** $\mathcal{O}$ can be generalized to Kac-Moody algebras (cd **Kac-Moody algebra**) (especially in the affine case), and the Kazhdan-Lusztig conjecture is true here as well by a result of M. Kashiwara and T. Tanisaki, [12]. There is another remarkable extension of the theory to a new class of objects, called **quantum groups**, which are deformations involving a parameter q of the hyperalgebra associated with $\mathfrak{g}$ (or, at least in type A, of the coordinate ring of G). The representation theory for $\mathfrak{g}$ extends to those algebras and there is a category $\mathcal{O}$ as well. In a series of papers [10, 11] D. Kazhdan and G. Lusztig produced an equivalence between a certain category $\mathcal{O}$ for affine Kac-Moody algebras and the category $\mathcal{O}$ of quantum groups, where the parameter q is specialized to a root of unity. As a consequence they showed that the Lusztig conjecture holds for quantum groups in characteristic 0 at roots of unity. But this case appears to be lying in between the characteristic 0 theory of category $\mathcal{O}$ and the characteristic p theory of representations of G. H.H. Andersen, J.C. Jantzen and W. Soergel found a (very elaborate) method to compare representations of quantum groups at p-th roots of unity with those of algebraic groups in characteristic p, which finally lead to their result [1].

The formal characters of the irreducible representations of quantum groups at roots of unity or equivalenty the decomposition multilpicities of irreducible modules in q-Weyl modules can be computed using character formulas for tilting modules, which are by definition modules which have a filtrations by Weyl- and induced modules. W. Soergel used the results of M. Kashiwara and T. Tanisaki and of D. Kazhdan and G. Lusztig to derive in [16, 17] character formulas for indecomposable tilting modules. Those give another basis of the Grothendieck group and W. Soergel's approach provides a much faster algorithm to compute the decomposition matrices of quantum groups at roots of unity in characteristic 0, (at least in the simply laced case and conjectuarally in general).

Computing the crystal basis of the fock space A. Lascoux, B. Leclerc and J.-Y. Thibon noticed an astonishing coincidence of their calculations with decomposition tables for general linear groups and symetric groups and conjectured in [13] that one can derive the decomposition matrices of quantum groups of type A over fields of characteristic 0 (or equivalently of q-Schur algebras, defined by R. Dipper and G. James as q-deformations of Schur algebra $S(n,r)$, see [6], [7] at roots of unity by comparing the standard and the crystal basis of fock space. This conjecture was extended and proved by S. Ariki in [2, 18]. The concrete computation is given here again by evaluating certain polynomials at one. It is remarkable that those have nonnegative integral coefficients. It is conjectured that these have a deeper meaning: They should give the composition multiplicities of the various layers in the Jantzen filtration of the q-Weyl modules.

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