Generalized q-Schur algebras and modular representation theory of finite groups with split (BN)-pairs

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Abstract

We introduce a generalized version of a q-Schur algebra (of parabolic type) for arbitrary Hecke algebras over extended Weyl groups. We describe how the decomposition matrix of a finite group with split BN-pair, with respect to a non-describing prime, can be partially described by the decomposition matrices of suitably chosen q-Schur algebras. We show that the investigated structures occur naturally in finite groups of Lie type.

Introduction

In a series of papers ([4, 5, 37, 14] Gordon James and the first named author used representations of Hecke- and of q-Schur algebras to derive a classification of the ℓ-modular irreducible representations of general linear groups $G = GL_n(q)$ for primes ℓ which are coprime to q. It turned out that the ℓ-decomposition matrices of G are completely determined by decomposition matrices of certain q-Schur algebras by an algorithm involving only combinatorics as partitions and the Littlewood-Richardson rule (see [14]).

The connection between the group algebra of G on the one side, and Hecke- and q-Schur algebras on the other side is given by certain functors H, called quotients of Hom-functors and their preinverses, which were investigated in [6, 7]. First Hecke algebras come up as endomorphism rings of certain induced modules for G, and these functors connect their representations with those of G. This part of the theory was extented to arbitrary finite groups of Lie type in [10, 11]. For general linear groups q-Schur algebras turn up as endomorphism rings of certain Hecke algebra modules. In fact quotients of Hom-functors

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can be defined here as well, one obtains a q-analogue of the classical Schur functors. The functors H are then used to show that the q-Schur algebras are isomorphic to endomorphism rings of certain G-modules as well, and that one can define again quotients of Hom-functors S, now connecting representations of q-Schur algebras and of G. In [28] Hiss and the second named author extended these results to finite classical groups in the special case of linear primes ℓ (about a third of the primes dividing the group order).

The main purpose of this paper is to exhibit how far these methods carry in the general case of finite groups G of Lie type for arbitrary primes ℓ different from the describing characteristic of the group. One of our main results is the extension of the second step, the q-Schur algebra approach in the situation of [10, 11]. In addition we shall show that the methods work in other cases too. Here a fundamental step has been done by Geck, Hiss and Malle, who showed that the endomorphism ring of Harish-Chandra induced cuspidal irreducible modules is always a Hecke algebra (see [24]), extending Howlett-Lehrer theory [32]. In addition they constructed corresponding quotients of Hom-functors.

However it should be pointed out that our results are not as nice and complete as in the case [14] of general linear or [28] of classical groups for linear primes: In general we can construct only a part of the irreducible representations of G and we get only partial information on the decomposition numbers in terms of generalized q-Schur algebras. It seems to be also clear, that the information provided by our results here is about everything what q-Schur algebras can give. To obtain information on the remaining irreducible representations of G and the corresponding decomposition numbers one probably needs fundamentally new methods.

We now describe briefly how we proceed in this paper. If G is a finite group with split BN-pair, one has the distribution of complex characters in so called Harish-Chandra series, HC-series for short. They are indexed by pairs whose first entry is a representative L of conjugacy classes of Levi subgroups of G, and the second entry is a cuspidal irreducible character of L. Attached to such an HC-series we have a Hecke algebra and one can define functors between the categories of G-modules and the module categories of the respective Hecke algebras.

In section one we first review the results on Hom-functors needed in the following. Then we define quotients of projective $\mathcal{O}G$-lattices by factoring out constituents which do not belong to a fixed HC-series. These quotients are defined by a functor which is compatible with HC-induction.

In section two we introduce the central object of the paper, projective restriction systems associated with HC-series. It turns out that this axiomatic setting enables us to step up from using Hecke- to using q-Schur algebras to obtain information on representations belonging to the corresponding HC-series. We investigate how modules, which are contained in such a projective restriction system, behave under Harisch-Chandra restriction. We exhibit in particular a close relation between the poset structure of Levi subgroups of G and module structure of the modules in the projective restriction system. As a special case one can bear in mind the Steinberg lattices of finite groups of Lie type.

In section three we then connect our results of section two and those of section one to get our main result on decomposition numbers.

In section four we consider the most important groups with split BN-pairs, the finite groups of Lie type. We show that projective restriction systems appear regularly for these groups and apply our results to obtain at the end a new concrete result for the unipotent characters of finite unitary groups.

This paper has a long history: A preliminary version was out already at the end of 1996 and it was referred to in several papers which meanwhile appeared. In between there were several improvements and generalisations, which may explain the delay in part, (however the discovery of several gaps in proofs, which had to be closed, contributed subtantially as well). The main results were announced at several conferences, in [12] and in [27].

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1 Preliminaries

Throughout G denotes a finite group. We let $\mathcal{O}$ be a complete discrete valuation ring. The quotient field of $\mathcal{O}$ is denoted by K and its residue field by k. We assume that k is of characteristic ℓ for some prime ℓ dividing the order of G. Moreover we take $(K, \mathcal{O}, k)$ to be an ℓ-modular splitting system for G. Thus both fields, K and k are splitting fields for all subgroups of G.

For $H \leq G$ we denote the normalizer of H in G by $\mathcal{N}_G(H)$.

Modules are, if not stated otherwise, right modules, and homomorphisms act from the opposite side. Let R be a commutative domain. An *R-order* A is an R-algebra which is free and finitely generated as R-module, and an *A-lattice* is an A-module which is finitely generated and free as R-module. The category of A-lattices is denoted by $\mathfrak{Lat}_A$ and of A-modules by $\mathfrak{Mod}_A$. Since R is commutative we may write the action of scalars on A-modules on the left assuming always that R acts centrally on every module.

If possible we omit tensor symbols. For instance, if B is some sublattice of A and M is a B-lattice we frequently write MA for the induced module $M \otimes_B A$. Or if $R = \mathcal{O}$, and if M is an A-lattice, KA denotes the finite dimensional K-algebra $K \otimes_{\mathcal{O}} A$, and KM the KA-module $K \otimes_{\mathcal{O}} M$. The k-Algebra $k \otimes_{\mathcal{O}} A$ is denoted by kA or by $\bar{A}$ and the $\bar{A}$-module $k \otimes_{\mathcal{O}} M$ by $\bar{M}$. Similarly, if H is a group and χ is a KH-character, its associated Brauer character, that is the restriction of χ to ℓ-regular classes of H, is denoted by $\bar{\chi}$.

The *radical* of an A-lattice M is the intersection of its maximal sublattices and is denoted by $\mathrm{Jac}(M)$. The *socle* $\mathrm{soc}(M)$ of M is the maximal completely reducible submodule of M and the *head* $\mathrm{hd}(M)$ of M is the maximal completely reducible factor module of M. Note that $\mathrm{hd}(M) = M/\mathrm{Jac}(M)$.

We need some results on quotients of Hom-functors from [6] and [7]. We begin with the basic setup there and supplement a few further results which will be needed later on.

Let T be a semiperfect R-algebra which is finitely generated as R-module. All occurring modules are, if not stated otherwise, finitely generated. Let $M \in \mathfrak{Mod}_T$, and let $\beta : P \to M$ be a projective presentation of M. Thus $P \in \mathfrak{Mod}_T$ is projective, and β is an epimorphism. Let $\mathcal{E} = \mathrm{End}_T(P)$. We take $\mathcal{E}_\beta = \{\phi \in \mathcal{E} \mid \phi(\ker \beta) \subseteq \ker \beta\}$. The endomorphism ring of M is denoted by $\mathcal{H}$. Obviously $J_\beta = \{\psi \in \mathcal{E} \mid \mathrm{im}\, \psi \leq \ker \beta\}$ is an ideal of $\mathcal{E}_\beta$ and $\mathcal{E}_\beta / J_\beta \cong \mathcal{H}$ as R-algebra canonically, (comp. [6, 2.1]). Our basic hypothesis is now the following:

Hypothesis 1.1 *For $M \in \mathfrak{Mod}_T$ we say that the projective presentation $\beta : P \to M$ of M satisfies* 1.1, *if $\mathcal{E}_\beta = \mathcal{E}$.*

Then J_β is an ideal of $\mathcal{E}$, and we identify $\mathcal{H}$ and $\mathcal{E}/J_\beta$ by the canonical isomorphism induced by β. We have now a functor

$$H = H^\beta = H_M^\beta : \mathfrak{Mod}_T \to \mathfrak{Mod}_{\mathcal{H}}$$

which takes the T-module V to the $\mathcal{H}$-module $\mathrm{Hom}_T(P,V)/\mathrm{Hom}_T(P,V)J_\beta$. On maps H^β is defined in the obvious way.

Notation 1.2 The *P-torsion submodule* $t_P(V)$ is the unique maximal submodule X of $V \in \mathfrak{Mod}_T$ with respect to the property that $\mathrm{Hom}_T(P,X) = (0)$. The *kernel* $\mathbf{ker}P$ of P is the full subcategory of $\mathfrak{Mod}_T$, whose objects are T-modules V with $\mathrm{Hom}_T(P,V) = (0)$. So $V \in \mathbf{ker}P$ precisely if $t_P(V) = V$. We have a functor $A_P : \mathfrak{Mod}_T \to \mathfrak{Mod}_T$ taking $V \in \mathfrak{Mod}_T$ to $V/t_P(V)$. We say that V is *P-torsionless* if $t_P(V) = (0)$. So $A_P(V)$ is the maximal P-torsionless factor module of V. We define the functor $\hat{H}_M = \hat{H} : \mathfrak{Mod}_{\mathcal{H}} \to \mathfrak{Mod}_T$ to be the composite functor $A_P \circ (- \otimes_{\mathcal{H}} M)$. For $V \in \mathfrak{Mod}_T$ the *trace* or *combined image* $\tau_P(V)$ of P in V is the T-submodule of V generated by the images of all homomorphisms $\phi : P \to V$.

The following lemma follows immediately from projectivity of P:

Lemma 1.3 *Let $P, V, U \in \mathfrak{Mod}_T$ and suppose that P is projective. Let $f : V \to U$ be T-linear. Then*

$$f(\tau_P(V)) \leq \tau_P(U) \quad \textit{and} \quad f(t_P(V)) \leq t_P(U).$$

Thus τ_P and t_P are really endofunctors of $\mathfrak{Mod}_T$.

It was shown in [6, 2.16] that $\hat{H}$ is a right inverse of the functor H. Thus in particular, if T is a finite dimensional algebra over some field, $\mathfrak{H}$ is a complete set of non-isomorphic irreducible $\mathcal{H}$-modules, and $\mathfrak{T}_M$ a complete set of non-isomorphic irreducible T-modules which are not taken to the zero module under the functor H, then H induces a bijection between $\mathfrak{T}_M$ and $\mathfrak{H}$ [6, 2.28].

More precisely we have the following theorem (see [7, 1.2]):

Theorem 1.4 *Suppose that $\beta : P \to M$ satisfies* 1.1. *Then*

(i) *If M is P-torsionless and X is a right ideal of $\mathcal{H}$. Then*

$$\hat{H}(X) = XM \cong A_P(X \otimes_{\mathcal{H}} M).$$

(ii) *If R is a field then $\mathfrak{T}_M$ is a complete set of nonisomorphic irreducible constituents of* $\mathrm{hd}(M)$. *Then every indecomposable direct summand of M has simple head and factoring out the Jacobson radical induces a bijection between the isomorphism classes of indecomposable direct summands of M and the elements of $\mathfrak{T}_M$.*

Let now T be an $\mathcal{O}$-order in the semisimple K-algebra KT. We say V is an *irreducible T-lattice* if $KV = K \otimes_{\mathcal{O}} V$ is an irreducible KT-module.

Recall that an kT-module V is *liftable*, if $V = k \otimes_{\mathcal{O}} U$ for some T-lattice U. Moreover, if U, V are T-lattices, then

$$\mathrm{Hom}_{KT}(KU, KV) \cong K \otimes_{\mathcal{O}} \mathrm{Hom}_T(U, V) \tag{1.5}$$

canonically. Similarly we have an homomorphism of k-spaces

$$\overline{\mathrm{Hom}_T(U, V)} = k \otimes_{\mathcal{O}} \mathrm{Hom}_T(U, V) \to \mathrm{Hom}_{\bar{T}}(\bar{U}, \bar{V}) \tag{1.6}$$

which is injective but not necessarily surjective. The homomorphisms in the image of that map are called *liftable*, and similarly $\mathrm{Hom}_{\bar{T}}(\bar{U}, \bar{V})$ is said to be *liftable*, if this homomorphism is bijective.

We take $M \in \mathfrak{Lat}_T$ and assume that the projective presentation $\beta : P \to M$ satisfies (1.1). By [6, 4.7] Hypothesis 1.1 (for $R = \mathcal{O}$) is equivalent to $\ker(1_K \otimes_{\mathcal{O}} \beta)$ having no irreducible constituent in common with KM. Moreover if this holds, the decomposition matrix of $\mathcal{H}$ is a submatrix of the decomposition matrix of T, [6, 4.10]. In [6, 4.7] it was shown that 1.1 holds then for $1_R \otimes_{\mathcal{O}} \beta$ for all choices of $R \in \{K, \mathcal{O}, k\}$. The three resulting Hom-functors and their right inverses are distinguished by a suffix R. We summarize:

Theorem 1.7 *Let $\beta : P \to M$ be a projective presentation of the T-lattice M and suppose that $K\ker(\beta)$ and KM have no irreducible constituent in common. Then $1_R \otimes_{\mathcal{O}} \beta$ satisfies Hypothesis* 1.1 *for all choices of $R \in \{K, \mathcal{O}, k\}$, the endomorphism ring of $\bar{M}$ is liftable and the decomposition matrix of $\mathcal{H} = \mathrm{End}_T(M)$ is part of the decomposition matrix of T.*

Lemma 1.8 *Let $\beta : P \to M$ be a projective presentation of the T-lattice M satisfying Hypothesis* 1.1*. Then RM is RP-torsionless for $R = K$ and $\mathcal{O}$.*

Recall that an algebra A over some field is called *quasi Frobenius*, if there exists a nondegenerate, associative bilinearform on A. This implies in particular, that the regular representation of A is injective, that is A is a self-injective algebra, and every homomorphism between two right ideals is given by left multiplication by some element of A.

If the form is in addition symmetric, we call A *symmetric*. For instance, group algebras over fields are symmetric algebras. Here is a generalisation to $\mathcal{O}$-orders:

Notation 1.9 An $\mathcal{O}$-order $\mathcal{H}$ is called *integrally quasi Frobenius* if there exists an associative bilinearform on $\mathcal{H}$ whose Gram determinant with respect to any basis of $\mathcal{H}$ is a unit in $\mathcal{O}$.

A sublattice V of some $\mathcal{O}$-lattice M is called *pure* if the factor module M/V is a torsion free (and hence free) $\mathcal{O}$-module. If $V \leq M$ then the intersection of all pure sublattices of M which contain V is the unique minimal pure sublattice containing V and is denoted by $\sqrt{V}$. One checks easily that $\sqrt{V} = KV \cap M$.

Note that the bilinearform on $\mathcal{H}$ induces nondegenerate associative bilinearforms on $R\mathcal{H}$ for $R = K$ and $R = k$, such that these algebras are quasi Frobenius. In our applications later on we shall deal with Hecke algebras which are known to be integrally quasi Frobenius (indeed integrally symmetric).

Here is a result, which we will need later. For a proof see [7, 1.30 and 1.36]:

Theorem 1.10 *Let $\beta : P \to M$ be a projective presentation of the T-lattice M satisfying Hypothesis* 1.1. *Let X and Y be pure right ideals of $\mathcal{H} = \mathrm{End}_T(M)$. Suppose that $\mathcal{H}$ is integrally quasi Frobenius. Denote the associated Hom-functor by $H = H^\beta$. Then*

(i) $H(XM) = H(\sqrt{XM}) = X$.

(ii) *Every homomorphism from $\sqrt{XM}$ to $\sqrt{YM}$ maps XM into YM. Restricting homorphisms induces an isomorphism*

$$\mathrm{Hom}_{RT}(R \otimes_{\mathcal{O}} \sqrt{XM}, R \otimes_{\mathcal{O}} \sqrt{YM}) \xrightarrow{\sim} \mathrm{Hom}_{RT}(XM, YM)$$

for $R = \mathcal{O}, K$.

(iii) *The functor H induces an isomorphism*

$$H_R : \mathrm{Hom}_{RT}(R \otimes_{\mathcal{O}} \sqrt{XM}, R \otimes_{\mathcal{O}} \sqrt{YM}) \xrightarrow{\sim} \mathrm{Hom}_{R\mathcal{H}}(RX, RY), \tag{1.11}$$

for $R = \mathcal{O}$ and K.

The following result is a refinement of [10, 3.2 and 3.3].

Lemma 1.12 *Let G be a finite group and let $(K, \mathcal{O}, k)$ be a split ℓ-modular system for G for some prime ℓ. Let P be a projective $\mathcal{O}G$-lattice and suppose that the irreducible KG-module M occurs with multiplicity one as constituent of KP. Let X be an $\mathcal{O}G$-lattice in M such that $KX = M$. Then*

$$\mathrm{Hom}_{\mathcal{O}G}(P, X) \cong \mathcal{O},$$

and for a generator φ of $\mathrm{Hom}_{\mathcal{O}G}(P, X)$ we have the following: The projective cover of the image $V = \mathrm{im}\,(\varphi)$ is an indecomposable direct summand Q of P. The simple kG-module $D = \bar{Q}/\mathrm{Jac}\bar{Q}$ is the head of $\bar{V}$, that is $\bar{V}/\mathrm{Jac}\bar{V} \cong D$, and D occurs as composition factor of $\bar{V}$ with multiplicity one. Moreover,

$$\mathrm{End}_{kG}(\bar{V}) \cong k \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}G}(V) \cong k$$

and the image U of $\bar{\varphi} = 1_k \otimes_{\mathcal{O}} \varphi : \bar{P} \to \bar{X}$ is an epimorphic image of $\bar{V}$ and $\bar{Q}$ is the projective cover of U. Thus U has simple head D occurring with multiplicity one as composition factor of U. In particular

$$\mathrm{End}_{kG}(U) \cong k.$$

Proof Let $P = P_1 \oplus \ldots \oplus P_m$ be a decomposition of P into a direct sum of projective indecomposable $\mathcal{O}G$-lattices. Then $KP = KP_1 \oplus \ldots \oplus KP_m$ and KX occurs as irreducible direct summand of precisely one KP_i, say of KP_1, and we set $Q = P_1$. Thus restricting maps induces an isomorphism

$$\mathrm{Hom}_{KG}(KP, KX) \cong \mathrm{Hom}_{KG}(KQ, KX),$$

where the left hand side is one dimensional. Since

$$\mathrm{Hom}_{KG}(KP, KX) \cong K \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}G}(P, X) \cong \mathrm{Hom}_{KG}(KQ, KX)$$

we conclude that
$$\mathrm{Hom}_{\mathcal{O}G}(P, X) \cong \mathcal{O} \cong \mathrm{Hom}_{\mathcal{O}G}(Q, X).$$
The generator φ of $\mathrm{Hom}_{\mathcal{O}G}(P, X)$ is a generator of $\mathrm{Hom}_{\mathcal{O}G}(Q, X)$ too, hence $\varphi : Q \to \mathrm{im}\,(\varphi) = V$ is the minimal projective cover of V. Being indecomposable $\bar{Q}$ has simple head $\mathrm{hd}(\bar{Q}) = \bar{Q}/\mathrm{Jac}(\bar{Q})$ and this must be the head of $\bar{V}$ as well. By Brauer's reciprocity law the multiplicity of D as composition factor of $\bar{X}$ equals the multiplicity of $M = KX$ as irreducible constituent of $KQ = KP_1$, and hence is one. Thus D occurs only once as composition factor of $\bar{V}$ as well. If $\wp$ denotes the generator of the unique maximal ideal $(\wp)$ of $\mathcal{O}$, then $k = \mathcal{O}/(\wp)$ and we have $\bar{V} \cong V/\wp V$, whereas the image U of $\bar{\varphi} = 1_k \otimes_{\mathcal{O}} \varphi : \bar{P} \to \bar{X}$ is given as $U = V/(V \cap \wp X)$. Since $\wp V \subseteq \wp X$, the module U is an epimorphic image of $\bar{V}$ and the remaining claims of the lemma follow. $\square$

Lemma 1.13 *Let G and $(K, \mathcal{O}, k)$ be as in the previous lemma. Let X be an $\mathcal{O}G$-lattice and suppose that its character χ decomposes $\chi = \chi_1 + \chi_2$, where χ_2 is the character of a pure sublattice M of X and χ_1 is an irreducible character afforded by $S = X/M$. Assume further that KS occurs in KP with multiplicity one, where P is the projective cover of S and let $D = \bar{S}/\mathrm{Jac}(\bar{S})$. Then D is an irreducible kG-module and occurs with multiplicity one in $\bar{S}$. Moreover*
$$\mathrm{End}_{kG}(\bar{S}) \cong k$$
and, if D does not occur as composition factor of M, then $X = S \oplus M$, and P is indecomposable.

Proof Let $\beta : P \to S$ be an epimorphism. Then Lemma 1.12 implies immediately that P is indecomposable with head $D = \bar{S}/\mathrm{Jac}(\bar{S})$ which is irreducible. Furthermore the multiplicity of D in $\bar{S}$ is one and $\mathrm{End}_{kG}(\bar{S}) \cong k$.

Consider the following diagram, where $\pi : X \to S$ is the canonical epimorphism:

$$\begin{tikzcd}
 & & & \ker\beta \arrow[d] & \ker\rho \arrow[dl] \\
 & & & P \arrow[dl, "\exists\rho"'] \arrow[d, "\beta"] & \\
(0) \arrow[r] & M \arrow[r] & X \arrow[r, "\pi"] & S \arrow[r] & (0)
\end{tikzcd}$$

By projectivity of P we find $\rho : P \to X$ such that $\pi\rho = \beta$. Let $U = \ker\beta$. Now $\pi\rho(U) = \beta(U) = (0)$, hence $\rho(U) \subseteq \ker\pi$. Tensoring over $\mathcal{O}$ by K yields
$$(1_K \otimes \rho)(KU) \subseteq \ker(1_K \otimes \pi) = KM.$$
We conclude that either KU and KM have a composition factor in common, or $KU \leq \ker(1_K \otimes \rho)$, and hence $U \leq \ker\rho$. Since D is not a composition factor of $\bar{M}$ we see,

using Brauer reciprocity, that KM and KP do not have a composition factor in common, therefore the same holds for $KU \leq KP$ and KM.

Thus $U \leq \ker \rho$. We have an induced map $\hat{\rho}$ from $P/U \cong S$ to X such that $\rho = \hat{\rho}\alpha$, where $\alpha : P \to P/U$ is the natural epimorphism. Similarly β induces an isomorphism $\hat{\beta}$ from $P/\ker\beta$ to S. We have $\pi\hat{\rho} = \hat{\beta}$, and therefore $\pi\hat{\rho}\hat{\beta}^{-1} = 1_S$ and π splits, as desired.

□

Let G be a finite group with split BN-pair of characteristic p. We want to investigate representations of G in the non-describing characteristic case, hence we assume that the prime p is different from ℓ. The notion of a BN-pair means that we have given a Borel subgroup B and the monomial subgroup N of G. For the definition and details on groups with BN-pair we refer to [2]. Occasionally, G has to be viewed as a Levi subgroup of a larger finite group with split BN-pair $\tilde{G}$. We denote the BN-pair of $\tilde{G}$ by $\tilde{B}$ and $\tilde{N}$ and say that $\tilde{G}$ has a $\tilde{B}\tilde{N}$-pair. In this situation we often assume that G is a *standard* Levi subgroup of $\tilde{G}$, that is, if $B = UT$ and $\tilde{B} = \tilde{T}\tilde{U}$ is the Levi decomposition of B and $\tilde{B}$ respectively, then $T = \tilde{T}$, $B \subseteq \tilde{B}$ and $B\tilde{U} = \tilde{B}$. As a consequence the monomial subgroup N of G, being the normalizer $\mathcal{N}_G(T)$, is contained in $\tilde{N}$ and $W = N/T \leq \tilde{N}/T = \tilde{W}$, indeed the *Weyl group* W of G is a standard parabolic subgroup of $\tilde{W}$ generated by a subset of the standard generators (simple reflections) of $\tilde{W}$.

The set of Levi subgroups of G will be denoted by $\mathcal{L}_G$ and for $L \in \mathcal{L}_G$, we denote by $\mathcal{L}_{G,L}$ the set of Levi subgroups of G containing L. Note that every Levi subgroup of G is conjugate to a standard Levi subgroup.

For $L \in \mathcal{L}_G$ and $R \in \{K, \mathcal{O}, F\}$, we define the functor *Harish-Chandra induction* R_L^G as follows: We choose a parabolic subgroup P of G such that L is a Levi complement of P. Let U be the Levi kernel, that is the unipotent radical, of P. Thus P is the semi direct product $P = U \rtimes L$. If M is an RL-lattice we may consider it as RP-lattice with trivial U-action. The corresponding functor is called *inflation* and is denoted by Inf_L^P. The functor R_L^G is now defined to be inflation followed by ordinary induction Ind_P^G. Thus for $M \in \mathfrak{Lat}_{RL}$ we have:

$$\mathrm{R}_L^G M = \mathrm{Ind}_P^G \circ \mathrm{Inf}_L^P(M).$$

Dually we define *Harish-Chandra restriction* T_L^G to be the restriction Res_P^G from RG-modules to RP-modules followed by Fix_U, which is the functor mapping the RP-module M to the RL-module $\mathrm{Fix}_U(M) = \{m \in M \mid mu = m \text{ for all } u \in U\}$. The fact that the order of U is invertible in R implies that R_L^G and T_L^G are adjoint functors, and the collection of functors R_L^G and their adjoints satisfy transitivity and a Mackey formula. They were investigated in a more general context in [10] and [8]. We point out that most of our general results in the following sections can be generalized to the situation in [8]. Morover, in this paper and independently in [33] it was shown that the functors R_L^G and their adjoints are independent of the choice of the parabolic subgroup P of which L is a Levi subgroup, provided the order of U is invertible in R. For fields R of characteristic 0 this result is known for a long time and was observed first by Deligne (comp. e.g. [42]). Of course these functors yield corresponding maps on characters and Brauer characters which again are denoted by R_L^G and T_L^G.

Harish-Chandra induction and restriction (or HC-induction and HC-restriction for short) provide the basic tools for the Harish-Chandra theory which subdivides the irreducible

RG-modules for $R = K$ or k into *Harish-Chandra series* (or HC-series for short). First an RL-module M is called *cuspidal* if $\mathrm{T}^L_{L'}(M) = (0)$ for every proper Levi subgroup L' of L. The irreducible RG-module V belongs to the HC-series $S(G/L, M)_R$ if V is a composition factor of the head (or alternatively of the socle) of $\mathrm{R}^G_L(M)$. If $R = K$ we usually omit the index K. The subgroup L and the irreducible cuspidal RL-module M are unique up to conjugation in G respectively in the normalizer $\mathcal{N}_G(L)$ of L in G. Analogously we define the HC-series $S(G/L, \chi)_R$ for the irreducible cuspidal (Brauer-) character χ of L. This partition of the irreducible RG-modules and characters into HC-series is well known for fields R of chracteristic 0 (see e.g. [2]), and was introduced by Hiss in [29] for fields R of positive characteristic ℓ. If the RG-module X is in the HC-series $S(G/L, M)$, the Levi subgroup L is called *semisimple vertex* and the irreducible cuspidal RL-module M *semisimple source* of X (see [8]). As remarked, those are unique up to conjugation in G and hence we may (and usually do) choose as representatives for semisimple vertices the unique standard Levi subgroups.

The following two lemmas are just formulated for the convenience of the reader. They can be easily derived from [29, 5.8] by Frobenius reciprocity and Mackey decomposition. Let now $L \leq M$ be Levi subgroups of G. We take an irreducible cuspidal kL-module X and an irreducible kG-module Y.

Lemma 1.14 *Y lies in the HC-series $S(G/L, X)_k$ if and only if we find $Z, Z' \in S(M/L, X)_k$ such that Y is isomorphic to some composition factor of* $\mathrm{hd}(\mathrm{R}^G_M Z)$ *and to some composition factor of* $\mathrm{soc}(\mathrm{R}^G_M Z')$.

Lemma 1.15 *Y lies in the series $S(G/L, X)_k$ if and only if there exist $Z, Z' \in S(M/L, X)_k$ such that Z is isomorphic to some composition factor of* $\mathrm{hd}(\mathrm{T}^G_M Y)$ *and Z' to some composition factor of* $\mathrm{soc}(\mathrm{T}^G_M Y)$.

Notation 1.16 Let $\mathfrak{X}$ be a set of irreducible characters of G and let Y be an $\mathcal{O}G$-lattice affording the character χ. We may write uniquely $\chi = \psi + \psi'$, where every irreducible constituent of ψ belongs to $\mathfrak{X}$ and no irreducible constituent of ψ' belongs to $\mathfrak{X}$. We find pure sublattices V and V' of Y such that V affords ψ and V' affords ψ'. We define $\pi_{\mathfrak{X}}(Y) := Y/V'$, the $\mathfrak{X}$-*quotient* of Y and $\iota_{\mathfrak{X}}(Y) := V$, the $\mathfrak{X}$-*sublattice* of Y. If $\mathfrak{X}$ consists of a single character ϕ say we write $\pi_\phi(Y)$ and $\iota_\phi(Y)$ instead of $\pi_{\{\phi\}}(Y)$ and $\iota_{\{\phi\}}(Y)$ respectively. Moreover, we set $\pi_{\mathfrak{X}}(\chi) = \psi = \iota_{\mathfrak{X}}(\chi)$. We define $\pi_{\mathfrak{X}}(\bar{Y}) := \overline{\pi_{\mathfrak{X}}(Y/V')}$ and $\iota_{\mathfrak{X}}(\bar{Y}) := \overline{\iota_{\mathfrak{X}}(Y/V')}$. We note that the latter two depend on Y and not just on $\bar{Y}$. Obviously $\pi_\phi(Y)$ is zero or affords a multiple of the character ϕ. Let $\mathfrak{C}_{\mathfrak{X}}$ be the full subcategory of the category of finitely generated $\mathcal{O}G$-lattices whose objects are those $Y \in \mathfrak{Lat}_{\mathcal{O}G}$ satisfying $\iota_{\mathfrak{X}}(Y) = Y$.

The construction of π is precisely, what is needed to apply the results from the beginning of this section, as the following results demonstrates. It follows immediately from the definitions in 1.16:

Lemma 1.17 *Let Y be a projective $\mathcal{O}G$-lattice and $\mathfrak{X}$ a set of irreducible characters of G. Let $M = \pi_{\mathfrak{X}}(Y)$ and $\beta : Y \to M$ be the natural projection. Then β satisfies Hypothesis* 1.1 *and the requirements of Theorem* 1.7. *In particular the decomposition matrix of the $\mathcal{O}$-order $\mathcal{H} = \mathrm{End}_{\mathcal{O}G}(M)$ is part of the ℓ-modular decomposition matrix of G.*

By construction π and ι are functorial: If we have given $Z \in \mathfrak{Lat}_{\mathcal{O}G}$ with $U = \pi_{\mathfrak{X}}(Z)$ and $U' = \iota_{\mathfrak{X}}(Z)$ and if $f : Y \to Z$, then $f|_V : V \to U$ and $f|_{V'} : V' \to U'$. Thus we have induced maps $\pi_{\mathfrak{X}}(f) : \pi_{\mathfrak{X}}(Y) \to \pi_{\mathfrak{X}}(Z)$ and $\iota_{\mathfrak{X}}(f) : \iota_{\mathfrak{X}}(Y) :\to \iota_{\mathfrak{X}}(Z)$. Thus $\pi_{\mathfrak{X}}$ and $\iota_{\mathfrak{X}}$ are funtors from $\mathfrak{Lat}_{\mathcal{O}G}$ onto $\mathfrak{C}_{\mathfrak{X}}$, indeed both are left inverses of the embedding of $\mathfrak{C}_{\mathfrak{X}}$ into $\mathfrak{Lat}_{\mathcal{O}G}$. Obviously for $\mathfrak{X} = \emptyset$ the functors $\pi_{\mathfrak{X}}$ and $\iota_{\mathfrak{X}}$ are both the zero functor, taking every module to the zero module. If $\mathfrak{X}$ contains all irreducible characters of G we get the identity functors.

Lemma 1.18 *Let $Y, Z \in \mathfrak{Lat}_{\mathcal{O}G}$. Let $\mathfrak{X}$ be a set of irreducible characters of G. If either Y or Z is in $\mathfrak{C}_{\mathfrak{X}}$ then the maps*

$$\pi_{\mathfrak{X}} : \mathrm{Hom}_{\mathcal{O}G}(Y, Z) \to \mathrm{Hom}_{\mathcal{O}G}(\pi_{\mathfrak{X}}(Y), \pi_{\mathfrak{X}}(Z))$$

and

$$\iota_{\mathfrak{X}} : \mathrm{Hom}_{\mathcal{O}G}(Y, Z) \to \mathrm{Hom}_{\mathcal{O}G}(\iota_{\mathfrak{X}}(Y), \iota_{\mathfrak{X}}(Z))$$

induced by the functors $\pi_{\mathfrak{X}}$ and $\iota_{\mathfrak{X}}$ are isomorphisms.

Proof This is immediate by tensoring by K and comparing dimensions. □

Lemma 1.18 does not carry over to liftable kG-modules of the form $\bar{Y}$ and $\bar{Z}$ for $Y, Z \in \mathfrak{Lat}_{\mathcal{O}G}$ kG-modules, since the canonical embedding $k \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}G}(Y, Z) \to \mathrm{Hom}_{kG}(\bar{Y}, \bar{Z})$ is not surjective in general. However it is a bijection in case that either Y or Z is a projective $\mathcal{O}G$-lattice (see e.g. [39, Theorem 14.7]).

Lemma 1.19 *Let $Y, Z \in \mathfrak{Lat}_{\mathcal{O}G}$ such that $\mathrm{Hom}_{kG}(\bar{Y}, \bar{Z})$ is liftable. Let $\mathfrak{X}$ be a set of irreducible characters of G and let $\mathfrak{C}_{\mathfrak{X}}$ be defined as in* 1.16*. Then we have:*

(i) *Let $Z \in \mathfrak{C}_{\mathfrak{X}}$. Then* $\mathrm{Hom}_{kG}(\bar{Y}, \bar{Z}) \cong \mathrm{Hom}_{kG}(\pi_{\mathfrak{X}}(\bar{Y}), \bar{Z})$.
(ii) *Let $Y \in \mathfrak{C}_{\mathfrak{X}}$. Then* $\mathrm{Hom}_{kG}(\bar{Y}, \bar{Z}) \cong \mathrm{Hom}_{kG}(\bar{Y}, \iota_{\mathfrak{X}}(\bar{Z}))$.

Proof By Definition 1.16 we have $\pi_{\mathfrak{X}}(\bar{Y}) = \bar{Y}/\bar{V}'$ for $V' \in \mathfrak{Lat}_{\mathcal{O}G}$ and $\iota_{\mathfrak{X}}(V') = (0)$. Denoting the projection from $\bar{Y}$ onto $\pi_{\mathfrak{X}}(\bar{Y})$ by $\bar{\pi}$ we have an embedding

$$\bar{\pi}^* : \mathrm{Hom}_{kG}(\pi_{\mathfrak{X}}(\bar{Y}), \bar{Z}) \to \mathrm{Hom}_{kG}(\bar{Y}, \bar{Z})$$

defined by $\bar{\pi}^*(f) = f \circ \bar{\pi}$. By assumption and Lemma 1.18 we have

$$\mathrm{Hom}_{kG}(\bar{Y}, \bar{Z}) = k \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}G}(Y, Z) \cong k \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}G}(\pi_{\mathfrak{X}}(Y), Z) \subseteq \mathrm{Hom}_{kG}(\pi_{\mathfrak{X}}(\bar{Y}), \bar{Z}),$$

hence, comparing dimensions, $\bar{\pi}^*$ is an isomorphism. ii) is proven analogously. □

We shall study now how the functors π and ι behave with respect to HC-induction and -restriction, where we take in 1.16 HC-series of characters of G as set $\mathfrak{X}$.

Notation 1.20 Let G be a group with split BN-pair which is a Levi subgroup of a group $\tilde{G}$ with split $\tilde{B}\tilde{N}$-pair. Let $\mathcal{M} \subset \mathcal{L}_G$ and let $\mathcal{C}$ be a set of cuspidal irreducible characters of elements of $\mathcal{M}$. We say the pair $(\mathcal{M}, \mathcal{C})$ is *closed under conjugation* in $\tilde{G}$ if $M \in \mathcal{M}$ implies $M^x \in \mathcal{M}$ and $\chi \in \mathcal{C}$ implies $\chi^x \in \mathcal{C}$ for all $x \in \tilde{G}$ such that $M^x \in \mathcal{L}_G$. If $M \in \mathcal{L}_G$ we set $\mathcal{M}_M := \mathcal{M} \cap \mathcal{L}_M$, and we define $\mathcal{C}_M$ to be the set of those KM'-characters χ with $M' \in \mathcal{M}_M$ and $\chi \in \mathcal{C}$. Note that $(\mathcal{M}_M, \mathcal{C}_M)$ is closed under conjugation in $\tilde{G}$ if $(\mathcal{M}, \mathcal{C})$ is. On the other hand if $(\mathcal{M}, \mathcal{C})$ is an arbitrary pair for $G \in \mathcal{L}_{\tilde{G}}$, then $(\mathcal{M}, \mathcal{C})$ is a pair for $\tilde{G}$. However, even if $(\mathcal{M}, \mathcal{C})$ is closed under conjugation by G, it is not necessarily closed under conjugation by $\tilde{G}$.

Now let $(\mathcal{M}, \mathcal{C})$ be fixed. We set for every $M \in \mathcal{L}_G$

$$\mathfrak{X}_M := \bigcup_{\substack{L \in \mathcal{M}_M, \\ \chi \in Irr(L) \cap \mathcal{C}_M}} S(M/L, \chi).$$

Then we define $\pi^M_{(\mathcal{M},\mathcal{C})} = \pi_{\mathfrak{X}_M}$. If no ambiguities can arise we omit the index $(\mathcal{M}, \mathcal{C})$ and write simply $\pi^M = \pi_{\mathfrak{X}_M}$.

Remark 1.21 *Obviously if M does not contain any $L \in \mathcal{M}$ then $\pi_{\mathfrak{X}_M} = \pi_\emptyset$ is the zero functor. Moreover it should be pointed out that the* restriction *π^M of π to $M \in \mathcal{L}_G$ essentially depends upon the choice of $(\mathcal{M}, \mathcal{C})$, not on the functor $\pi = \pi^G$ alone. For instance we choose $x \in G$ such that $L \leq M$ and $L^x \leq M$ are not conjugate in $M \in \mathcal{L}_G$, and we take an irreducible cuspidal character χ of L. Then χ^x is an irreducible cuspidal character of L^x. We set $\mathcal{M} = \{L\}$, $\mathcal{C} = \{\chi\}$, $\mathcal{M}^x = \{L^x\}$ and $\mathcal{C}^x = \{\chi^x\}$. Then obviously*

$$\pi^G_{(\mathcal{M},\mathcal{C})} = \pi^G_{(\mathcal{M}^x,\mathcal{C}^x)}$$

but

$$\pi^M_{(\mathcal{M},\mathcal{C})} \not\cong \pi^M_{(\mathcal{M}^x,\mathcal{C}^x)}.$$

To apply Results 1.16 and 1.19 we need the following lemma which follows immediately from Mackey decomposition (compare [10, 1.14]) and from transitivity of HC-induction:

Lemma 1.22 *Let ψ be a KG-character.*

(i) *Let $M \in \mathcal{L}_{G,L}$ and assume that all constituents of ψ belong to the HC-series $S(G/L, \chi)$, where χ is an irreducible cuspidal KL-character. Then every constituent of $\mathrm{T}^G_M \psi$ belongs to a series $S(M/L^x, \chi^x)$ for some $x \in G$ such that $L^x \leq M$.*

(ii) *Suppose $G \in \mathcal{L}_{\tilde{G}}$ and assume that every irreducible constituent of ψ belongs to some HC-series $S(G/L^x, \chi^x)$, where $x \in \tilde{G}$ such that $L^x \leq G$. Then $\mathrm{R}^{\tilde{G}}_G \psi$ belongs to $S(\tilde{G}/L, \chi)$.*

Proof To prove (i) we may assume that ψ is irreducible. Then ψ is constituent of $\mathrm{R}^G_L \chi$ by assumption and hence $\mathrm{T}^G_M \psi$ is a summand of

$$\mathrm{T}^G_M \mathrm{R}^G_L \chi = \sum_x \mathrm{R}^M_{L^x \cap M} \mathrm{T}^{L^x}_{L^x \cap M} \chi^x,$$

where x runs through a set of double coset representatives of parabolic subgroups having M and L as Levi complements, by [10, 1.14]. But these double coset representatives can be chosen in N, (see [2, 2.1]). Moreover $\mathrm{T}^{L^x}_{L^x\cap M}\chi^x = (0)$ unless $L^x \cap M = L^x$, that is $L^x \leq M$ by cuspidality of χ. Thus the result follows. Part (ii) is shown similarly, observing that for $x \in N$ we have $\mathrm{R}^{\tilde{G}}_{L^x} \cong \mathrm{R}^{\tilde{G}}_{L}$ by [33] respectively [9, 5.2]. □

Note that in the lemma above we may take M and L to be standard Levi subgroups of G, and G to be a standard Levi subgroup of $\tilde{G}$ (by adjusting G, then M and then L by conjugating by an element of $\tilde{G}$,G and M respectively). In this case we may take x to be an element of N or in (ii) of $\tilde{N}$. Now Lemma 1.22 tells us that under HC-restriction we might get terms with semi simple HC-vertex in M^x, where $x \in N$ such that $L^x \leq M$. We therefore define

Definition 1.23 Let $M \in \mathcal{L}_{G,L}$. Then we set

$$N_{L\subseteq M} = \{x \in G | L^x \leq M\}.$$

The letter N stands here for the monomial group of G, since we can always assume that L is standard in M and M in G, and then choose x to be an element of N. Thus for instance

$$\tilde{N}_{L\subseteq G} = \{x \in \tilde{G} \mid L^x \subseteq G\}.$$

The following Corollary follows now easily from Lemma 1.22:

Corollary 1.24 *Suppose that $(\mathcal{M},\mathcal{C})$ is closed under conjugation in G and let $M \in \mathcal{L}_G$. Then we have isomorphisms of functors*

(i) *For $(\mathcal{M},\mathcal{C})$ as in* 1.20 *and $M \in \mathcal{L}_G$ we have*

$$\mathrm{R}^G_M \pi^M \cong \pi^G \mathrm{R}^G_M$$
$$\mathrm{R}^G_M \iota^M \cong \iota^G \mathrm{R}^G_M.$$

(ii)

$$\mathrm{T}^G_M \pi^G \cong \pi^M \mathrm{T}^G_M$$
$$\mathrm{T}^G_M \iota^G \cong \iota^M \mathrm{T}^G_M.$$

Proof First note that it suffices to check that the functors agree on characters, since R^G_M and T^G_M preserve purity of sublattices. We show the second part of Corollary, the proof of the first being similar. So let ψ be an irreducible KG-character in $S(G/L,\chi)$, and $L \in \mathcal{M}$. If $\psi \in \mathcal{C}$ then $\pi^G(\psi) = \psi$ and $\pi^G(\psi) = (0)$ otherwise. If $L^x \not\leq M$ for all $x \in G$, then $\mathrm{T}^G_M\psi = (0)$ by Mackey decomposition and then

$$0 = \mathrm{T}^G_M \pi^G(\psi) = \pi^M(\mathrm{T}^G_M\psi)$$

in all cases. Suppose $L^x \leq M$. Since $S(G/L,\chi) = S(G/L^x,\chi^x)$ we may assume $x = 1$, that is $L \leq M$. Let $\chi \in \mathcal{C}$ then $\chi^x \in \mathcal{C}_M$ for all $x \in N_{L\subseteq M}$ and hence

$$\pi^M(\mathrm{T}^G_M\psi) = \mathrm{T}^G_M\psi = \mathrm{T}^G_M\pi^G(\psi).$$

If $\chi \notin \mathcal{C}$ then $\chi^x \notin \mathcal{C}$ for all $x \in N_{L\subseteq M}$ and we get

$$0 = \pi^M(\mathrm{T}_M^G\psi) = \mathrm{T}_M^G(0) = \mathrm{T}_M^G\pi^G(\psi).$$

The other isomorphisms of functors are shown similarly $\square$

Corollary 1.25 *Suppose that $(\mathcal{M}, \mathcal{C})$ is closed under conjugation in G and let $M \in \mathcal{L}_G$. Suppose $Y \in \mathfrak{Lat}_M$ is projective. Set $X = \pi^M(Y)$ and consider*

$$\beta : \mathrm{R}_M^G Y \to R_M^G X,$$

where $\beta = \mathrm{R}_M^G\psi$ setting $\psi : Y \to X$ to be the natural projection. Then β satisfies the requirements of Theorem 1.7. In particular the decomposition matrix of $\mathcal{H} = \mathrm{End}_{\mathcal{O}G}(\mathrm{R}_M^G(X))$ is part of the ℓ-modular decomposition matrix of G.

Proof By Corollary 1.24

$$\pi^G(\mathrm{R}_M^G Y) = \mathrm{R}_M^G\pi^M(Y) = \mathrm{R}_M^G X.$$

Thus the result follows from 1.17 applied to the projective $\mathcal{O}G$-module $\mathrm{R}_M^G Y$. $\square$

2 Projective restriction systems

We introduce now *projective restriction systems* for finite groups with split BN-pairs. First we describe the set up which we shall be dealing with:

Notation 2.1 Let G be a finite group with split BN-pair. We fix $L \in \mathcal{L}_G$ and an irreducible cuspidal KL-character χ_L. We assume that χ_L occurs with multiplicity one as an irreducible constituent of the character afforded by some projective $\mathcal{O}L$-lattice Y_L. Thus we may apply Lemmas 1.12 and 1.13.

If $G \in \mathcal{L}_{\tilde{G}}$ for some group $\tilde{G}$ with $\tilde{B}\tilde{N}$-pair such that $B = \tilde{B} \cap G$ and $N = \tilde{N} \cap G$, we take in 1.20

$$\mathcal{M} := \{L^x \mid x \in \tilde{N}_{L\subseteq G}\} \quad \text{and} \quad \mathcal{C} := \{\chi_L^x \mid x \in \tilde{N}_{L\subseteq G}\}.$$

For $M \in \mathcal{L}_G$ we write

$$\pi^M := \pi^M_{(\mathcal{M},\mathcal{C})}.$$

In particular for $M = G$ we omit the superscript G and write $\pi = \pi^G$. A similar notation is used for ι.

Finally we set

$$X_L := \pi^L(Y_L), \quad X_L' := \iota^L(Y_L), \quad \text{and} \quad D_L := \bar{X}_L/\mathrm{Jac}(\bar{X}_L).$$

Remark 2.2 *By Lemma* 1.13, $D_L \cong \mathrm{hd}(\bar{X}_L)$ *is irreducible and is the head of the projective cover* $\bar{Y}_1$ *of* $\bar{X}_L$ *which is isomorphic to a direct summand of* $\bar{Y}_L$. *If* Y_1 *denotes its lift to an* $\mathcal{O}L$*-sublattice of* Y_L, *which exists by projectivity of* Y_L, *then* $\iota^L(Y_L) \subseteq Y_1$. *Since* kL *is a symmetric algebra, the head* D_L *of* $\bar{Y}_1$ *and its socle are isomorphic. Thus the lattice* X'_L *is different from* X_L *unless* $\bar{X}_L$ *is irreducible, as* D_L *is the head of* $\bar{X}_L$ *and the socle of* $\bar{X}'_L$. *Obviously we have*

$$\mathrm{Hom}_{kL}(\bar{X}_L, \bar{X}'_L) \cong k.$$

Definition 2.3 Keep the notation introduced in 2.1. In view of Lemma 1.12 we may assume that Y_L is indecomposable and hence is the projective cover of X_L. Note that this implies that $\bar{Y}_L$ is an principal indecomposable kL-module corresponding to the irreducible module D_L. In particular D_L is the head $\bar{Y}_L/\mathrm{Jac}(\bar{Y}_L)$ of $\bar{Y}_L$. Suppose in addition that χ_G is an irreducible constituent of multiplicity one in $\mathrm{R}_L^G \chi_L$ such that the following holds:

(i) There exists a projective $\mathcal{O}G$-lattice Y_G such that χ_G is the character of

$$\pi^G(Y_G) =: X_G.$$

Again we may (and do) assume that Y_G is indecomposable.

(ii) For $M \in \mathcal{L}_{G,L}$ let $\mathrm{T}_M^G \chi_G = \psi_1 + \ldots + \psi_n$ be a decomposition of $\mathrm{T}_M^G \chi_G$ into irreducible constituents. Let X_i be an $\mathcal{O}M$-lattice affording ψ_i for $i = 1, \ldots, n$. Then for $i \neq j$, the kM-modules $\bar{X}_i$ and $\bar{X}_j$ have no composition factor in common.

We set $X_M := \pi_{(L,\chi_L)}(\mathrm{T}_M^G X_G)$ for every $M \in \mathcal{L}_{G,L}$ and $X'_M := \iota_{(M,\chi_M)}(\mathrm{R}_L^M X_L)$. We denote the characters of X_M by χ_M and the projective cover of X_M by Y_M.

The *projective restriction system* to (L, χ_L) then consists of the data $\{X_M, Y_M, X'_M \mid M \in \mathcal{L}_{G,L}\}$ and is denoted by $\mathcal{PR}(X_G, Y_L)$.

Remark 2.4 *Our notation is consistent, since part* (i) *of* 2.3 *implies*

$$\pi(Y_G) = \pi^G_{(\mathcal{M},\mathcal{C})}(Y_G) = \pi_{(L,\chi_L)}(Y_G) = X_G.$$

Hence $X_G = \pi_{(L,\chi_L)}(\mathrm{T}_G^G X_G)$ *since* X_G *is irreducible and* $\chi_G \in S(G/L, \chi_L)$ *by assumption. Similarly* Y_L *is the projective cover of* $\pi_{(L,\chi_L)} \mathrm{T}_L^G X_G = X_L$, *since*

$$\mathrm{Hom}_{\mathcal{O}L}(\mathrm{T}_L^G X_G, X_L) \cong \mathrm{Hom}_{\mathcal{O}G}(X_G, \mathrm{R}_L^G X_L) \cong \mathcal{O}$$

by Frobenius reciprocity. Moreover, if $x \in \mathcal{N}_G(L)$ *is such that the conjugate character* χ_L^x *is different from* χ_L, *then* χ_L^x *cannot be a constituent of* Y_L. *With other words we have* $\pi^L(Y_L) = X_L$. *Otherwise the simple head of* $\bar{Y}_L$ *is composition factor of* $\bar{X}_L$ *as well as of* $\bar{X}_L^x$, *since* Y_L *is indecomposable. But* X_L^x *affords* χ_L^x, *and it is constituent of* $\mathrm{T}_L^G X_G$, *hence* $\bar{X}_L^x$ *has by* (ii) *of* 2.3 *no composition factor with* $\bar{X}_L$ *in common. We also note that part* (ii) *is particularly satisfied if the summands of* $\mathrm{T}_M^G \chi_G$ *are in pairwise different* ℓ*-modular blocks of* $\mathcal{O}M$. *This will be the fact which we prove in concrete applications.*

For the remainder of this section we fix a projective restriction system $\mathcal{PR}(X_G, Y_L)$. Throughout $M \in \mathcal{L}_{G,L}$. The next lemma is trivial:

Lemma 2.5 *Let $x \in \tilde{G}$, then the conjugate projective restriction system*

$$\mathcal{PR}^x(X_G, Y_L) = \mathcal{PR}(X_G^x, Y_L^x)$$

is the projective restriction system given as $\{X_M^x, Y_M^x, X_M^{x\prime} \mid M \in \mathcal{L}_{G,L}\}$.

Observe that X_M^x is the conjugate module M^x-module $(X_M)^x$. We have to distinguish this from certain other modules coming up in the conjugate restriction system:

Notation 2.6 *Let $x \in \tilde{G}$ and $M \in \mathcal{L}_{G^x,L^x}$. Then the M-module*

$$X_{M,x} = \pi_{(L^x,\chi_L^x)}(\mathrm{T}_M^G X_G^x)$$

and its projective cover $Y_{M,x}$ belong to the conjugate restriction system $\mathcal{PR}^x(X_G, Y_L)$. In particular, if $M \in \mathcal{L}_{G,L}$ and $x \in G$ is such that $L^x \subseteq M$, then

$$\mathcal{PR}^x(X_G, Y_L) = \mathcal{PR}(X_G, Y_L^x)$$

and

$$X_{M,x} = (X_{M^{x^{-1}}})^x = \pi_{(L^x,\chi_L^x)}(\mathrm{T}_M^G X_G).$$

We denote the related terms similarly: For example $\chi_{M,x}$ denotes the character afforded by $X_{M,x}$.

In the following we shall use frequently the Mackey formula for HC-induction and restriction in connection with the property of X_L to be cuspidal. In the Mackey formula we deal with double coset representatives $\mathcal{D}_{LM}$ of parabolic subgroups containing L respectively $M \in \mathcal{L}_{G,L}$ as Levi complement. If L and M are standard Levi subgroups we may choose $\mathcal{D}_{LM}$ to be *distinguished* by general theory, that is it is contained in the monomial subgroup N of G as set of preimages of the distinguished double coset representatives of the corresponding standard parabolic subgroups of the Weyl group $W = N/T$ of G, (for details see e.g. [2]). This choice then ensures, that the intersections $M \cap L^x$ with $x \in \mathcal{D}_{LM}$ are again standard Levi subgroups of G. Since X_L is cuspidal by assumption, all terms in the Mackey formula

$$\mathrm{T}_M^G \mathrm{R}_L^G X_L = \bigoplus_{x \in \mathcal{D}_{LM}} \mathrm{R}_{M \cap L^x}^M \mathrm{T}_{M \cap L^x}^{L^x} X_L^x \tag{2.7}$$

with $L^x \nsubseteq M$ are zero, that is we have to sum only over $x \in N_{L\subseteq M} \cap \mathcal{D}_{LM}$. But $x \in G$ is contained in $N_{L\subseteq M}$ precisely if $LxM = xM$. If in addition $x \in \mathcal{D}_{LM}$, then $L^x = L^x \cap M \in \mathcal{L}_G$. Obviously M acts on $N_{L\subseteq M}$ by right translation, where the orbit of $x \in N_{L\subseteq M}$ under this action is the double coset $xM = LxM$.

Observe that $\mathcal{N}_G(L) \subseteq N_{L\subseteq M}$, and that this group acts on $N_{L\subseteq M}$ by left translation. We take $\mathcal{N}_{M,L}^G$ to be a set of representatives of $\mathcal{N}_G(X_L) \times M$-orbits on $N_{L\subseteq M}$ (containing $1 \in N_{L\subseteq M}$), where $\mathcal{N}_G(X_L)$ is the stabilizer of X_L in $\mathcal{N}_G(L)$. Note that each such orbit contains elements of $\mathcal{D}_{LM}$ and we choose therefore $\mathcal{N}_{M,L}^G$ to be a subset of our selected set of double coset representatives $\mathcal{D}_{LM}$. If we need to emphasize the dependence of the chosen object X_L we write $\mathcal{N}_{M,L}^G(X_L)$, and we define similar representatives for other L-modules as for example $\mathcal{N}_{M,L}^G(D_L)$ etc.

It follows now that $\{L^x \mid x \in \mathcal{N}^G_{M,L}\}$ is a set of Levi subgroups of M which for $x \neq 1$ are conjugate to L in G but not in M. Moreover $L^x = L^y$ with $x, y \in \mathcal{N}^G_{M,L}$ and $x \neq y$, implies $X^x_L \ncong X^y_L$.

Theorem 2.8 *Let $M \in \mathcal{L}_{G,L}$ and $x \in \mathcal{N}^G_{M,L}$. Then the $\mathcal{O}M$-lattice $X_{M,x}$ affords the character $\chi_{M,x}$ which is irreducible and occurs with multiplicity one as constituent of the character afforded by $\mathrm{T}^G_M(Y_G)$. Moreover*

$$\mathrm{T}^G_M X_G = \bigoplus_{x \in \mathcal{N}^G_{M,L}} X_{M,x}.$$

Proof The character χ_G of X_G is contained in the HC-series $S(G/L, \chi_L)$. Hence by Lemma 1.22 every irreducible constitutent of $\mathrm{T}^G_M \chi_G$ belongs to some $S(M/L^x, \chi^x_L)$ for some $x \in N_{L \subseteq M}$. We may assume that $x \in \mathcal{N}^G_{M,L}$. By assumption χ_G occurs with multiplicity one in $\mathrm{R}^G_L \chi_L$, hence by Lemma 2.5 χ_G occurs with multiplicity one in $\mathrm{R}^G_{L^x} \chi^x_L$ too. Therefore by Frobenius reciprocity and transitivity of HC-restriction

$$K \cong \mathrm{Hom}_{KL^x}(KX^x_L, \mathrm{T}^G_{L^x} KX_G) \cong \mathrm{Hom}_{KM}(\mathrm{R}^M_{L^x} KX^x_L, \mathrm{T}^G_M KX_G),$$

and consequently $\mathrm{T}^G_M \chi_G$ has precisely one constituent in $S(M/L^x, \chi^x_L)$ and this is by definition given as

$$\pi_{L^x, \chi^x_L}(\mathrm{T}^G_M \chi_G) = \chi_{M,x}.$$

In particular $\chi_{M,x}$ is an irreducible character and, being in different HC-series of M, we have $\chi_{M,x} \neq \chi_{M,y}$ for $x \neq y$ in $\mathcal{N}^G_{M,L}$. We have shown so far that

$$\mathrm{T}^G_M \chi_G = \sum_{x \in \mathcal{N}^G_{M,L}} \chi_{M,x}.$$

Being an $\mathcal{O}M$-lattice in an irreducible KM-module, $X_{M,x}$ is indecomposable. Note that X_G is an epimorphic image of Y_G and hence $X_{M,x}$, being an epimorphic image of $\mathrm{T}^G_M X_G$, is an epimorphic image of the projective $\mathcal{O}M$-lattice $\mathrm{T}^G_M Y_G$. Hence the projective cover $Y_{M,x}$ of $X_{M,x}$ is a direct summand of $\mathrm{T}^G_M Y_G$. In particular $\chi_{M,x}$ occurs as constituent of the character of $\mathrm{T}^G_M Y_G$. Remark 2.4 in conjunction with Lemma 2.5 implies

$$X_G = \pi^G(Y_G) = \pi_{(L,\chi_L)}(Y_G) = \pi_{(L^x,\chi^x_L)}(Y_G).$$

By Frobenius reciprocity, Corollary 1.24 and Lemma 1.18 we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}M}(\mathrm{T}_M^G Y_G, X_{M,x}) &\cong \mathrm{Hom}_{\mathcal{O}G}(Y_G, \mathrm{R}_M^G X_{M,x}) \\
&\cong \mathrm{Hom}_{\mathcal{O}G}(Y_G, \mathrm{R}_M^G \pi^M X_{M,x}) \\
&\cong \mathrm{Hom}_{\mathcal{O}G}(Y_G, \pi^G \mathrm{R}_M^G X_{M,x}) \\
&\cong \mathrm{Hom}_{\mathcal{O}G}(Y_G, \mathrm{R}_M^G X_{M,x}) \\
&\cong \mathrm{Hom}_{\mathcal{O}G}(Y_G, \pi^G \mathrm{R}_M^G (X_{M^{x^{-1}}})^x) \\
&\cong \mathrm{Hom}_{\mathcal{O}G}(Y_G, \pi^G \mathrm{R}_{M^{x^{-1}}}^G X_{M^{x^{-1}}}) \\
&\cong \mathrm{Hom}_{\mathcal{O}G}(Y_G, \pi^G \mathrm{R}_M^G X_M) \\
&\cong \mathrm{Hom}_{\mathcal{O}G}(\pi^G Y_G, \pi^G \mathrm{R}_M^G X_M) \\
&\cong \mathrm{Hom}_{\mathcal{O}G}(X_G, \pi^G \mathrm{R}_M^G X_M) \\
&\leq 1
\end{aligned}$$

We conclude that the multiplicity of $\chi_{M,x}$ in the character of $Y_{M,x}$ is precisely one. But now part (ii) of Definiton 2.3 states exactly that the hypothesis of Lemma 1.13 are satisfied for $\mathrm{T}_M^G X_G$, $X_{M,x}$ and the projective cover $Y_{M,x}$ of $X_{M,x}$. Thus $X_{M,x}$ is a direct summand of $\mathrm{T}_M^G X_G$ for all $x \in \mathcal{N}_{M,L}^G$ and the theorem follows. □

Note that Theorem 2.8 states in particular that KX_M is irreducible. Result 1.12 now implies:

Corollary 2.9 *Let $M \in \mathcal{L}_{G,L}$ and $x \in \mathcal{N}_{M,L}^G$. Then $D_{M,x} := \bar{X}_{M,x}/\mathrm{Jac}(\bar{X}_{M,x})$ is an irreducible kM-module occurring with multiplicity one as composition factor of $\bar{X}_{M,x}$. Moreover*

$$\dim_k(\mathrm{End}_{kM}(\bar{X}_M)) = 1,$$

and $\bar{Y}_{M,x}$ is the projective cover of $D_{M,x}$. In particular $\bar{Y}_{M,x}$ and hence $Y_{M,x}$ is indecomposable.

Note that

$$D_{M,x} = \mathrm{hd}(\bar{X}_{M,x}) = \mathrm{hd}((\bar{X}_{M^{x^{-1}}})^x) = (D_{M^{x^{-1}}})^x. \tag{2.10}$$

As a consequence of Theorem 2.8 and part (ii) of Definition 2.3 we get the following two results:

Corollary 2.11 *Let $M, M' \in \mathcal{L}_{G,L}$ with $M' \leq M$.*

(i) *Then $\pi_{(L,\chi_L)}^{M'}(\mathrm{T}_{M'}^M X_M) = X_{M'}$. In fact*

$$\mathrm{T}_{M'}^M X_M = \bigoplus_{y \in \mathcal{N}_{M',\,L}^M} X_{M',y}.$$

(ii) *Let V be an indecomposable projective kM'-module such that $V/\mathrm{Jac}(V)$ is a composition factor of $\bar{X}_{M'}$. Then*

$$\mathrm{Hom}_{kM'}(\mathrm{T}^{M}_{M'}\bar{X}_M, V) \cong \mathrm{Hom}_{kM'}(\bar{X}_{M'}, V)$$

and

$$\mathrm{Hom}_{kM'}(V, \mathrm{T}^{M}_{M'}\bar{X}_M) \cong \mathrm{Hom}_{kM'}(V, \bar{X}_{M'}).$$

Corollary 2.12 *Let $M \in \mathcal{L}_{G,L}$. Then*

$$\mathcal{PR}_M := \mathcal{PR}(X_M, Y_L) = \{X_{M'}, Y_{M'}, X'_{M'} \mid M' \in \mathcal{L}_{M,L}\}$$

is a projective restriction system to (L, χ_L).

We now investigate the projective cover Y_G of X_G:

Theorem 2.13 *Let $M \in \mathcal{L}_{G,L}$. Then*

$$\mathrm{T}^G_M Y_G = \bigoplus_{x \in \mathcal{N}^G_{M,L}} Y_{M,x} \oplus Z,$$

where Z is a projective $\mathcal{O}M$-lattice with $\pi^M(Z) = (0)$. Moreover,

$$\pi_{(L^x, \chi_L^x)}(\mathrm{T}^G_M Y_G) = X_{M,x},$$

and

$$\pi_{(L^x,\chi_L^x)}(Y_{M,y}) = \begin{cases} \pi_{(L^x,\chi_L^x)}(\mathrm{T}^G_M Y_G) = X_{M,x} & \text{if } x = y \\ (0) & \text{otherwise,} \end{cases}$$

where $x, y \in \mathcal{N}^G_{M,L}$. Thus in particular

$$\pi^M_{(\mathcal{M},\mathcal{C})}(Y_{M,x}) = \pi_{(L^x,\chi_L^x)}(Y_{M,x}) = X_{M,x}.$$

Proof Since $\mathrm{T}^G_M X_G$ is an epimorphic image of $\mathrm{T}^G_M Y_G$, the latter contains the projective cover of the former as a direct summand, hence by Theorem 2.8

$$\mathrm{T}^G_M Y_G = \bigoplus_{x \in \mathcal{N}^G_{M,L}} Y_{M,x} \oplus Z,$$

for some projective $\mathcal{O}M$-lattice Z. Now by Results 1.24 and 2.8

$$\begin{aligned}\bigoplus_{x \in \mathcal{N}^G_{M,L}} X_{M,x} \oplus \pi^M(Z) &= \bigoplus_{x \in \mathcal{N}^G_{M,L}} \pi^M(Y_{M,x}) \oplus \pi^M(Z) \\ &= \pi^M(\mathrm{T}^G_M Y_G) = \mathrm{T}^G_M X_G \\ &= \bigoplus_{x \in \mathcal{N}^G_{M,L}} X_{M,x} = \bigoplus_{x \in \mathcal{N}^G_{M,L}} \pi_{(L^x,\chi_L^x)}(\mathrm{T}^G_M X_G).\end{aligned}$$

Hence $\pi^M(Z) = (0)$ and

$$X_{M,x} = \pi_{(L^x,\chi_L^x)}(\mathrm{T}_M^G Y_G) = \pi_{(L^x,\chi_L^x)}(Y_{M,x}).$$

□

Corollary 2.14 *Let $M \in \mathcal{L}_{G,L}$. Then*

$$\pi_{(L^x,\chi_L^x)}(\mathrm{R}_M^{\tilde{G}} Y_M) = \mathrm{R}_M^{\tilde{G}} X_M,$$

for every $x \in \tilde{G}$.

Proof We only need to observe that on $\mathcal{O}\tilde{G}$-lattices $\pi^{\tilde{G}} = \pi^{\tilde{G}}_{(\mathcal{M},\mathcal{C})} = \pi_{(L^x,\chi_L^x)}$ for every $x \in \tilde{G}$, since L^x and L are conjugate in $\tilde{G}$. Thus by Lemma 1.24

$$\begin{aligned}\pi_{(L,\chi_L)}(\mathrm{R}_M^{\tilde{G}} Y_M) &= \pi^{\tilde{G}}(\mathrm{R}_M^{\tilde{G}} Y_M)\\ &= \mathrm{R}_M^{\tilde{G}} \pi^M Y_M\\ &= \mathrm{R}_M^{\tilde{G}} X_M\end{aligned}$$

□

Notation 2.15 Let $L \in \mathcal{L}_{G,L}$. In Theorem 2.8 we have seen that $\bar{X}_M$ has a unique maximal submodule, hence $D_M := \bar{X}_M/\mathrm{Jac}(\bar{X}_M)$ is irreducible. Now D_M can be cuspidal or not. We denote the subset of $\mathcal{L}_{G,L}$ of all M such that D_M is cuspidal by $\mathcal{L}^c(X_G, Y_L)$. Note that $L \in \mathcal{L}^c(X_G, Y_L)$ since χ_L and hence $D_L = \bar{X}_L/\mathrm{Jac}(\bar{X}_L)$ is cuspidal. Note too, that for $x \in G$ the set $\mathcal{L}^c(X_G, Y_L^x)$ consists precisely of the subgroups $M^x \in \mathcal{L}_{G,L^x}$ with $M \in \mathcal{L}^c(X_G, Y_L)$. In particular, $M \in \mathcal{L}^c(X_G, Y_L)$ does not imply $M \in \mathcal{L}^c(X_G, Y_L^x)$ in general, hence if D_M is cuspidal and $x \in N_{L\subseteq M}$ the kM-module $D_{M,x}$ is not necessarily cuspidal. However the irreducible kM^x-module $D_M^x = (D_M)^x$ is cuspidal in this case.

We fix now $M \in \mathcal{L}^c(X_G, Y_L)$ and $G' \in \mathcal{L}_{G,M}$. Note that $G' \in \mathcal{L}_{G,L}$, and that $N_{M\subseteq G'} \subseteq N_{L\subseteq G'}$. Take $x \in N_{M\subseteq G'}$. The $\mathcal{N}_G(X_L) \times G'$-orbit $\mathcal{N}_G(X_L)xG'$ of x in $N_{L\subseteq G'}$ is denoted by $[x]_{G',X_L}$ and we may assume that $x \in \mathcal{N}^G_{G',L}$. Thus $\bar{X}_{G',x}$ is one of the indecomposable direct summands of $\mathrm{T}_{G'}^G \bar{X}_G$ by Theorem 2.8.

Theorem 2.16 *Keep the notation introduced above. Then there is precisely one composition factor D_{G',M^x} of $\mathrm{T}_{G'}^G \bar{X}_G$ in the HC-series $S(G'/M^x, D_M^x)_k$ and it is in fact a composition factor of the direct summand $\bar{X}_{G',x}$ of $\mathrm{T}_{G'}^G \bar{X}_G$. Moreover, if $V \leq \mathrm{T}_{G'}^G \bar{X}_{G,x}$, then D_{G',M^x} is a composition factor of V if and only if D_M^x is a composition factor of $T_{M^x}^{G'} V$.*

Proof Using Frobenius reciprocity, part (ii) of Definition 2.3, and Theorem 2.8, we have

$$\begin{aligned}\operatorname{Hom}_{kG'}(\mathrm{T}_{G'}^{G}\bar{X}_G, \mathrm{R}_{M^x}^{G'}D_M^x) &= \operatorname{Hom}_{kM^x}(\mathrm{T}_{M^x}^{G}\bar{X}_G^x, D_M^x)\\ &= \bigoplus_{y\in\mathcal{N}_{M^x,L^x}^{G}} \operatorname{Hom}_{kM^x}(\bar{X}_{M^x,y}^x, D_M^x)\\ &= \operatorname{Hom}_{kM^x}(\bar{X}_M^x, D_M^x) \cong k,\end{aligned}$$

since in the last direct sum above the only nonzero term occurs for $y = 1$, again by part (ii) of Definition 2.3, (applied to the projective restriction system $\mathcal{PR}^x(X_G, Y_L)$). We conclude that $\mathrm{T}_{G'}^{G}\bar{X}_G$ and the socle of $\mathrm{R}_{M^x}^{G'}D_M^x$ have a common composition factor. But this is a composition factor D_{G',M^x} of $\mathrm{T}_{G'}^{G}\bar{X}_G$ in $S(G'/M^x, D_M^x)_k$.

Now let $V \leq \mathrm{T}_{G'}^{G}\bar{X}_G^x$. Then the embedding $V \to \mathrm{T}_{G'}^{G}\bar{X}_G^x$ induces an embedding of $\operatorname{Hom}_{kG'}(\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x, V)$ into $\operatorname{Hom}_{kG'}(\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x, \mathrm{T}_{G'}^{G}\bar{X}_G^x)$, and we have by Theorem 2.8, using 2.3, Frobenius reciprocity, and in addition the fact that Y_M is projective:

$$\begin{aligned}\operatorname{Hom}_{kM^x}(\bar{Y}_M^x, \mathrm{T}_{M^x}^{G'}V) &\cong \operatorname{Hom}_{kG'}(\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x, V)\\ &\leq \operatorname{Hom}_{kG'}(\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x, \mathrm{T}_{G'}^{G}\bar{X}_G^x)\\ &\cong \operatorname{Hom}_{kM^x}(\bar{Y}_M^x, \mathrm{T}_{M^x}^{G}\bar{X}_G^x)\\ &\cong \bigoplus_{y\in\mathcal{N}_{M^x,L^x}^{G}} \operatorname{Hom}_{kM^x}(\bar{Y}_{M^x}^x, \bar{X}_{M^x,y}^x)\\ &\cong \operatorname{Hom}_{kM^x}(\bar{Y}_M^x, \bar{X}_M^x)\\ &\cong k\otimes_{\mathcal{O}} \operatorname{Hom}_{\mathcal{O}M^x}(Y_M^x, X_M^x) \cong k\otimes_{\mathcal{O}}\mathcal{O} \cong k.\end{aligned} \tag{2.17}$$

Here we wrote $\bar{X}_{M^x,y}^x$ for the construction 2.6 applied to the conjugate module $\bar{X}_M^x$.

The first isomorphism in Equation 2.17 implies that there is a non-trivial homomorphism from $\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x$ to V if and only if D_M^x is a composition factor of $\mathrm{T}_{M^x}^{G'}V$. But in this case

$$\operatorname{Hom}_{kG'}(\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x, V) \cong \operatorname{Hom}_{kG'}(\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x, \mathrm{T}_{G'}^{G}\bar{X}_G).$$

We observe that $\bar{Y}_M^x$ is the projective cover of D_M^x, hence the projective cover of $\mathrm{R}_{M^x}^{G'}D_M^x$ is a direct summand of $\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x$ which in turn contains the projective cover $\bar{Y}_{G',M^x}$ of D_{G',M^x}, since this is contained in the HC-series $S(G'/M^x, D_M^x)_k$ and hence is an irreducible summand of $\operatorname{hd}(\mathrm{R}_{M^x}^{G'}D_M^x)$. Thus a nonzero homomorphism from $\mathrm{R}_{M^x}^{G'}\bar{Y}_M^x$ into $\mathrm{T}_{G'}^{G}\bar{X}_G^x$ (which exists and is unique up to scalar multiplies by 2.17), is nonzero on $\bar{Y}_{G',M^x}$. From this the last assertion of the theorem follows easily.

It remains to show that D_{G',M^x} is a composition factor of $\bar{X}_{G',x}$. To do this we apply Theorem 2.8 and the fact that homomorphisms from projective modules over k are liftable to $\mathcal{O}$ and obtain:

$$\mathcal{O} \cong \operatorname{Hom}_{\mathcal{O}G'}(\mathrm{R}_{M^x}^{G'}Y_M^x, T_{G'}^{G}X_G) \cong \bigoplus_{y\in\mathcal{N}_{G',L}^{G}} \operatorname{Hom}_{\mathcal{O}G'}(\mathrm{R}_{M^x}^{G'}Y_M^x, X_{G',y}),$$

hence there is precisely one $y \in \mathcal{N}^G_{G',L}$ such that

$$\operatorname{Hom}_{\mathcal{O}G'}(\mathrm{R}^{G'}_{M^x} Y^x_M, X_{G',y}) \neq (0). \tag{2.18}$$

We tensor by K and use 1.20, Corollary 1.24, Lemma 1.18 and Frobenius reciprocity to get:

$$\begin{aligned}
\operatorname{Hom}_{KG'}(\mathrm{R}^{G'}_{M^x} KY^x_M, KX_{G',y}) &= \operatorname{Hom}_{KG'}(\pi^{G'} \mathrm{R}^{G'}_{M^x} KY^x_M, KX_{G',y}) \\
&= \operatorname{Hom}_{KG'}(\mathrm{R}^{G'}_{M^x} \pi^{M^x} KY^x_M, KX_{G',y}) \\
&= \operatorname{Hom}_{KG'}(\mathrm{R}^{G'}_{M^x} KX^x_M, KX_{G',y}) \\
&= \operatorname{Hom}_{KM^x}(KX^x_M, \mathrm{T}^{G'}_{M^x} KX_{G',y}) \\
&\leq \operatorname{Hom}_{KM^x}(\mathrm{R}^{M^x}_{L^x} KX^x_L, \mathrm{T}^{G'}_{M^x} KX_{G',y}) \\
&= \operatorname{Hom}_{KL^x}(KX^x_L, T^{G'}_{L^x} KX_{G',y}).
\end{aligned} \tag{2.19}$$

Now $\pi_{(L^x,\chi^x_L)}(X_{G',y}) \neq (0)$ if and only if $y = x$, since $X_{G',y} = \pi_{(L^x,\chi^x_L)}(\mathrm{T}^G_{G'} X_G)$. Thus the first Hom-set in Formula 2.19 is zero for $y \neq x$, and hence nonzero for $x = y$ by Formula 2.18. We conclude that

$$\operatorname{Hom}_{kM^x}(\bar{Y}^x_M, \mathrm{T}^{G'}_{M^x} \bar{X}_{G',x}) = k \otimes_{\mathcal{O}} \operatorname{Hom}_{\mathcal{O}G'}(\mathrm{R}^{G'}_{M^x} Y^x_M, X_{G',x}) \neq (0).$$

and therefore D_{G',M^x} is a composition factor of $\bar{X}_{G',x}$ by the first part of the proof applied to $V = \bar{X}_{G',x}$. □

The previous theorem yields a unique composition factor D_{G',M^x} of $\bar{X}_{G',x}$ from the HC-series $S(G'/M^x, D^x_{M^x})$, for every $x \in \mathcal{N}^G_{G',L}$ such that $[x]_{G',L} \cap N_{M \subseteq G'}$ is not empty.

Let $x, y \in N_{M \subseteq G'}$. Then $S(G'/M^x, D^x_{M^x}) = S(G'/M^y, D^y_{M^y})$ if and only if there is a $g \in G'$ such that $M^{xg} = M^y$ and $D^{xg}_{M^{xg}} \cong D^y_{M^y}$. This means that x and y are in the same $\mathcal{N}(D_M) \times G'$-orbit of $N_{M \subseteq G'}$, where $\mathcal{N}(D_M) = \operatorname{stab}_{\mathcal{N}_G(M)}(D_M)$. Note that $\mathcal{N}(D_M)$ acts indeed on $N_{M \subseteq G'}$ by left translation since $\mathcal{N}_G(M)$ does. On the other hand, Theorem 2.16 tells us that every direct summand of $\mathrm{T}^G_{G'} \bar{X}_G$ contains at most one composition factor from one HC-series $S(G'/M^x, D^x_M)$, and all such series occur, and we have shown:

Lemma 2.20 *Keep the notation of Theorem 2.16. Let $x \in \mathcal{N}^G_{G',L}$. Then $[x]_{G',X_L} \cap N_{M \subseteq G'}$ is either empty or a union of $\mathcal{N}(D_M) \times G'$-orbits in $N_{M \subseteq G'}$.*

For general linear groups we have $\mathcal{N}(D_M) \leq \mathcal{N}(X_L)$ in the relevant cases, and the result is trivially true. We do not know if this holds in general. Theorem 2.16 implies now immediately:

Corollary 2.21 *Suppose $[x]_{G',X_L} \cap N_{M \subseteq G'}$ is empty. Then no composition factor of $X_{G',x}$ lies in an HC-series $S(G'/M^y, D_{M^y})$ for some $y \in N_{M \subseteq G'}$. If $[x]_{G',X_L} \cap N_{M \subseteq G'}$ is the disjoint union of the $\mathcal{N}(D_M) \times (G' \cap N)$-orbits of $y_1, \dots, y_m \in N_{M \subseteq G'}$,*

$$[x]_{G',X_L} \cap N_{M \subseteq G'} = \bigcup_{i=1}^m [y_i]_{G',D_M},$$

then for every $1 \leq i \leq m$ there is precisely one composition factor $D_{G',M^{y_i}}$ of $X_{G',x}$ in the HC-series $S(G'/M^{y_i}, D^{y_i}_M)$.

In the special case $G = G'$ we have obviously

$$\mathcal{N}^G_{G,M}(D_M) = \mathcal{N}^G_{G,L} = G, \tag{2.22}$$

hence we have precisely one orbit

$$[1]_{G,X_L} = [1]_{G,D_M} = G. \tag{2.23}$$

This implies:

Corollary 2.24 *Let $M \in \mathcal{L}^c(X_G, Y_L)$. Then X_G has a unique composition factor $D_{G,M}$ in $S(G/M, D_M)$.*

The next theorem provides information how the composition factor $D_{G,M}$ behaves under HC-restriction:

Theorem 2.25 *Let $M \in \mathcal{L}^c(X_G, Y_L)$ and $M \leq G' \in \mathcal{L}_{G,L}$. Then*

$$\mathrm{T}^G_{G'} D_{G,M} = \bigoplus_{x \in \mathcal{N}^G_{G',M}(D_M)} D_{G',M^x}.$$

In particular, $\mathrm{T}^G_{G'} D_{G,M}$ is semisimple. Moreover, if C is any composition factor of $\bar{X}_G$ not isomorphic to some $D_{G,M}$ then $\mathrm{T}^G_{G'} C$ has no composition factor equal to some D_{G',M^x} for $x \in \mathcal{N}^G_{G',M}(D_M)$.

Proof By definition we have $D_{G,M} \in S(G/M, D_M)_k$. Hence, by Lemma 1.15, we find irreducible kG'-modules $Z, Z' \in S(G'/M, D_M)_k$ such that Z is in the socle and Z' is in the head of $\mathrm{T}^G_{G'} D_{G,M}$.

By Theorem 2.16, $\mathrm{T}^G_{G'} \bar{X}_G$ has a unique composition factor in $S(G'/M, D_M)_k$, namely $D_{G',M}$. Now $D_{G,M}$ is a composition factor of $\bar{X}_G$ hence $\mathrm{T}^G_{G'} D_{G,M}$ is a subfactor of $\mathrm{T}^G_{G'} \bar{X}_G$. Thus every composition factor of $\mathrm{T}^G_{G'} D_{G,M}$ is a composition factor of $\mathrm{T}^G_{G'} \bar{X}_G$. We conclude that $\mathrm{T}^G_{G'} D_{G,M}$ has precisely one composition factor in $S(G'/M, D_M)_k$, namely $D_{G',M}$ and we have shown $Z = Z' = D_{G',M}$. Being in the head and in the socle of $\mathrm{T}^G_{G'} D_{G,M}$, the kG'-module $D_{G',M}$ splits off.

Applying Corollary 2.21 we get similarly that $D_{G',M^x} \in S(G'/M^x, D^x_M)$ splits of $\mathrm{T}^G_{G'} D_{G,M}$ for every $x \in \mathcal{N}^G_{G',M}(D_M)$. Thus

$$\mathrm{T}^G_{G'} D_{G,M} = \bigoplus_{x \in \mathcal{N}^G_{G',L}(D_M)} D_{G',M^x} \oplus Y,$$

where Y is some kG'-module, whose composition factors do not belong to any of the series $S(G'/M^x, D^x_M)$, $x \in N_{M \subseteq G'}$. But since $D_{G,M} \in S(G/M, D_M)$, every irreducible summand of the socle and the head of $\mathrm{T}^G_{G'} D_{G,M}$, hence of the socle and the head of Y, belongs to one of these HC-series. Therefore $Y = (0)$.

We see in particular that the composition factors of $\mathrm{T}^G_{G'} \bar{X}_G$ from these series are already composition factors of the subfactor $\mathrm{T}^G_{G'} D_{G,M}$ of $\mathrm{T}^G_{G'} \bar{X}_G$ and the last assertion also follows. $\square$

Remark 2.26 *Corollary* 2.21 *provides the recipe how the direct summands* D_{G',M^y} *of* $\mathrm{T}_{G'}^G D_{G,M}$ *divide up into composition factors of the direct summands* $X_{G',x}$ *of* $\mathrm{T}_{G'}^G X_G$*, where* y *runs through* $\mathcal{N}_{G',M}^G(D_M)$ *and* x *through* $\mathcal{N}_{M,L}^G$.

Theorem 2.27 *Let* $M, G' \in \mathcal{L}_{G,L}$ *such that* $M \leq G'$*. Then every homomorphism from* $\mathrm{R}_M^{G'} Y_M$ *to* $X_{G'}$ *factors through* $\mathrm{R}_M^{G'} X_M$ *and*

$$\mathrm{Hom}_{\mathcal{O}G'}(\mathrm{R}_M^{G'} X_M, X_{G'}) \cong \mathcal{O} \cong \mathrm{Hom}_{\mathcal{O}G'}(X_{G'}, \mathrm{R}_M^{G'} X_M).$$

If $\phi = \phi_{G',M}$ *in the first and* $\rho = \rho_{G',M}$ *in the second* Hom*-space are chosen such that every other homomorphism is a scalar multiple, then the following holds, writing* $1_k \otimes_{\mathcal{O}} \phi = \bar{\phi}$ *and* $1_k \otimes_{\mathcal{O}} \rho = \bar{\rho}$:

(i) *The kernel of* $\bar{\rho}$ *has no composition factor isomorphic to* $D_{G',M'}$ *for* $M' \in \mathcal{L}^c(X_{G'}, Y_L)$.
(ii) *The image* U *of* $\bar{\phi}$ *has simple head and contains* $D_{G',M'}$ *as composition factor for* $M' \in \mathcal{L}^c(X_{G'}, Y_L)$ *if and only if* M' *is conjugate in* G' *to a subgroup of* M.
(iii) *If in addition* $M \in \mathcal{L}^c(X_{G'}, Y_L)$ *then* $D_{G',M}$ *is the head* $U/\mathrm{Jac}(U)$ *of* $U = \mathrm{im}\,(\bar{\phi})$*. Moreover, if* $V \leq \bar{X}_{G'}$ *and* $D_{G',M}$ *is a composition factor of* V *and* $M' \leq M$, $M' \in \mathcal{L}^c(X_{G'}, Y_L)$*, then* $D_{G',M'}$ *is a composition factor of* V.

Proof In view of Result 2.12 we may assume that $G = G'$. Now by Lemma 1.18 every homomorphism from $\mathrm{R}_M^G Y_M$ to X_G factors through $\pi_{(L,\chi_L)}(\mathrm{R}_M^G Y_M)$. But if $\mathcal{M}'$ denotes the set of G-conjugates of L and $\mathcal{C}'$ the set of G-conjugate characters of χ_L, then $(\mathcal{M}', \mathcal{C}')$ is closed under conjugation in G. On the other hand on $\mathcal{O}G$-lattices $\pi_{(\mathcal{M}',\mathcal{C}')} = \pi_{(L,\chi_L)}$ and hence by Corollary 1.24 and Theorem 2.13

$$\pi_{(L,\chi_L)}(\mathrm{R}_M^G Y_M) = \pi_{(\mathcal{M}',\mathcal{C}')}(\mathrm{R}_M^G Y_M) = \mathrm{R}_M^G \pi_{(\mathcal{M}',\mathcal{C}')}^M (Y_M) = \mathrm{R}_M^G(\pi_{(L,\chi_L)}(Y_M)) = \mathrm{R}_M^G(X_M).$$

By Frobenius reciprocity and Theorem 2.8

$$\mathrm{Hom}_{\mathcal{O}G}(R_M^G X_M, X_G) \cong \bigoplus_{x \in \mathcal{N}_{M,L}^G} \mathrm{Hom}_{\mathcal{O}M}(X_M, X_{M,x}) \cong \mathrm{Hom}_{\mathcal{O}M}(X_M, X_M) \cong \mathcal{O}, \tag{2.28}$$

since $KX_{M,x} \ncong KX_M$ for $1 \neq x \in \mathcal{N}_{M,L}^G$. Similarly

$$\mathrm{Hom}_{\mathcal{O}G}(X_G, R_M^G X_M) \cong \mathcal{O}. \tag{2.29}$$

We take now ϕ and ρ to correspond to $1 \in \mathcal{O}$ in Equation 2.28 and 2.29 respectively. We first prove part (ii). By Frobenius reciprocity we have

$$\begin{aligned} k \cong \mathrm{Hom}_{kM}(\bar{Y}_M, \mathrm{T}_M^G U) &\cong \mathrm{Hom}_{kG}(\mathrm{R}_M^G \bar{Y}_M, U) \\ &\cong \mathrm{Hom}_{kG}(\mathrm{R}_M^G \bar{Y}_M, \bar{X}_G) \\ &\cong \mathrm{Hom}_{kG}(\mathrm{R}_M^G \bar{X}_M, \bar{X}_G) \\ &\cong \mathrm{Hom}_{kM}(\bar{Y}_M, \bar{X}_M) \end{aligned} \tag{2.30}$$

which shows in particular that D_M is a composition factor of $\mathrm{T}_M^G U$ and hence $\bar{X}_M$ is a submodule of $\mathrm{T}_M^G U$. Thus $D_{M'}$ is a composition factor of $\mathrm{T}_{M'}^G U \geq \mathrm{T}_{M'}^M \bar{X}_M$ for every

$M' \in \mathcal{L}^c(X_G, Y_L)$ with $M' \leq M$ and hence $D_{G,M'}$ is a composition factor of U by Theorem 2.16 (applied in the special case $G = G'$). Since all these composition factors have multiplicity one as composition factors of $\bar{X}_G$, the same holds for the submodule U of $\bar{X}_G$. Observe that the projective kG-module $\mathrm{R}_M^G \bar{Y}_M$ contains the projective cover Z of U as direct summand, and this is the projective cover of $\mathrm{hd}(U)$ as well. Since $\mathrm{Hom}_{kG}(\mathrm{R}_M^G \bar{Y}_M, U)$ is one dimensional, the same is true for $\mathrm{Hom}_{kG}(Z, U)$ and $\mathrm{hd}(U)$ must be simple.

On the other hand let $M' \in \mathcal{L}^c(X_G, Y_L)$ and assume that $D_{G,M'}$ is a composition factor of U. Then it is a composition factor of $\mathrm{R}_M^G \bar{X}_M$ since $\bar{\phi}$ maps this module onto U. Now $D_{G,M'}$ is in the HC-series $S(G/M', D_{M'})$ and therefore a composition factor of $\mathrm{hd}(\mathrm{R}_{M'}^G D_{M'})$. Thus $\mathrm{R}_{M'}^G \bar{Y}_{M'}$ contains the projective cover of $D_{G,M'}$ and there is a nonzero homomorphism from $\mathrm{R}_{M'}^G \bar{Y}_{M'}$ to $\mathrm{R}_M^G \bar{X}_M$. We have:

$$\begin{aligned}(0) &\neq \mathrm{Hom}_{kG}(\mathrm{R}_{M'}^G \bar{Y}_{M'}, \mathrm{R}_M^G \bar{X}_M) \\ &\cong \mathrm{Hom}_{kM'}(\bar{Y}_{M'}, \mathrm{T}_{M'}^G \mathrm{R}_M^G \bar{X}_M) \\ &\cong \bigoplus_{z \in \mathcal{D}_{M,M}} \mathrm{Hom}_{kM'}(\bar{Y}_{M'}, \mathrm{R}_{M' \cap M^z}^{M'} \mathrm{T}_{M' \cap M^z}^{M^z}(\bar{X}_{M^z}^z)).\end{aligned} \tag{2.31}$$

But the head of $\bar{Y}_{M'}$ is $D_{M'}$ and hence cuspidal by assumption. Thus such a homomorphism can exist only if $M' \leq M^z$ for some $z \in G$ or equivalently $z^{-1} \in N_{M' \subseteq M}$. Applying Frobenius reciprocity to 2.31 again and conjugating we conclude that

$$\mathrm{Hom}_{kM'^{z^{-1}}}(\bar{Y}_{M'^{z^{-1}}}, \mathrm{T}_{M'^{z^{-1}}}^M \bar{X}_M) \neq (0).$$

Setting $M'^{z^{-1}} = \tilde{M}$ we have by Corollary 2.8

$$\begin{aligned}\mathrm{Hom}_{k\tilde{M}}(\bar{Y}_{\tilde{M}}, \mathrm{T}_{\tilde{M}}^M \bar{X}_M) &= \bigoplus_{x \in \mathcal{N}_{\tilde{M},L}^M} \mathrm{Hom}_{k\tilde{M}}(\bar{Y}_{\tilde{M}}, \bar{X}_{\tilde{M},x}) \\ &= \mathrm{Hom}_{k\tilde{M}}(\bar{Y}_{\tilde{M}}, \bar{X}_{\tilde{M}}) = k.\end{aligned}$$

Thus we have $D_{G,\tilde{M}} \cong D_{G,M'}$ since $\tilde{M}$ and M' are conjugate in G and (ii) is shown.

To prove (iii) we observe that $D_{G,M}$ is a composition factor of U by part (ii). Suppose $D_{G,M}$ is not the head of U then there is a proper submodule V of U such that $D_{G,M}$ is a composition factor of V. As in 2.30 we show that $\mathrm{Hom}_{kM}(\bar{Y}_M, \mathrm{T}_M^G V) \neq (0)$, thus $\mathrm{im}\,\bar{\phi} \leq V$, a contradiction and therefore (iii) holds.

We now prove (i). First observe that

$$\mathrm{R}_M^G \bar{X}_M \cong \bigoplus_x \bar{X}_M \otimes x,$$

where x runs through a set of coset repesentatives of a parabolic subgroup of G having Levi complement M. Let $\tau : \mathrm{R}_M^G \bar{X}_M \to \bar{X}_M$ be the projection onto the summand $\bar{X}_M = \bar{X}_M \otimes 1$. By general theory the map $\hat{\rho}$ defined to be the restriction of $\tau \circ \bar{\rho}$ to $T_M^G \bar{X}$ is KM-linear and is the map which corresponds to ρ under the isomorphism

$$\mathrm{Hom}_{kG}(\bar{X}_G, \mathrm{R}_M^G \bar{X}_M) \cong \mathrm{Hom}_{kM}(\mathrm{T}_M^G \bar{X}_G, \bar{X}_M) \tag{2.32}$$

In particular $\hat{\rho}$ is non-zero, indeed it is an epimorphism. By Theorem 2.8 and transitivity of HC-induction we have

$$\mathrm{T}_L^M \mathrm{T}_M^G \bar{X}_G = \mathrm{T}_L^G \bar{X}_G = \bigoplus_{x \in \mathcal{N}_{L,L}^G} \bar{X}_{L,x},$$

in particular $\bar{X}_L$ occurs in $\mathrm{T}_L^G \bar{X}_G$ and hence in $\mathrm{T}_L^M(\mathrm{T}_M^G \bar{X}_G)$ exactly once as a direct summand. Of course the same holds for $\bar{X}_M$, that is $\bar{X}_L$ occurs in $\mathrm{T}_L^M \bar{X}_M$ exactly once. Suppose that the unique subspace $\bar{X}_L$ of $\bar{X}_G$ is in the kernel of $\bar{\rho}$ then it is also in the kernel of $\tau \circ \bar{\rho} = \hat{\rho}$. But we may take $\hat{\rho}$ to be the projection of

$$T_M^G \bar{X}_G = \bigoplus_{x \in \mathcal{N}_{M,L}^G} \bar{X}_{M,x}$$

onto $\bar{X}_{M,1} \cong \bar{X}_M$ and therefore the restriction $\tilde{\rho}$ of $\hat{\rho}$ to $\bar{X}_{M,1}$ is the identity map. Thus

$$\mathrm{T}_L^M \tilde{\rho} : \mathrm{T}_L^M \bar{X}_{M,1} \to \bar{X}_M$$

maps the unique subspace $\bar{X}_L$ of $\mathrm{T}_M^G \bar{X}_G$ into $\bar{X}_L \leq \mathrm{T}_L^M \bar{X}_M \leq \bar{X}_M$. Thus $\bar{X}_L$ can not be contained in the kernel of $\hat{\rho} = \tau \circ \bar{\rho}$ and hence not in the kernel of $\bar{\rho}$. Suppose $D_{G,M}$ is a composition factor of $\ker(\bar{\rho}) \leq X_G$. Then by (iii) $D_{G,M'}$ is composition factor of $\ker(\bar{\rho})$ for every $M' \in \mathcal{L}(X_G, Y_L)$ with $M' \leq M$. But $L \in \mathcal{L}^c(X_G, Y_L)$ by 2.15, hence $D_{G,L}$ is a composition factor of $\ker(\bar{\rho})$. Thus we have a nonzero homomorphism from the projective cover of $D_{G,L}$, which is a direct summand of $\mathrm{R}_L^G \bar{Y}_L$ into $\ker(\bar{\rho})$, hence by Frobenius reciprocity from $\bar{Y}_L$ into $\mathrm{T}_L^G \ker(\bar{\rho}) \leq \mathrm{T}_L^G \bar{X}_G$. But $\mathrm{Hom}_{kL}(\bar{Y}_L, \mathrm{T}_L^G \bar{X}_G)$ is one dimensional, and its image is the direct summand $\bar{X}_L$ of $\mathrm{T}_L^G \bar{X}_G$. We conclude that $\bar{X}_L$ is contained in $\mathrm{T}_L^G \ker(\bar{\rho})$ and hence in $\ker(\bar{\rho})$, a contradiction. This proves (i) and the theorem is shown. □

Corollary 2.33 *Let $G' \in \mathcal{L}_{G,L}$. Then the head $D_{G'}$ of $\bar{X}_{G'}$ is isomorphic to $D_{G',M}$ for some maximal $M \in \mathcal{L}^c(X_G, Y_L)$ with $M \leq G'$.*

Proof In view of Result 2.12 we may again assume that $G = G'$. If D_G is cuspidal we are done, since then $G \in \mathcal{L}^c(X_G, Y_L)$ is maximal, and D_G is the head of $\bar{X}_G$.

Assume $G \notin \mathcal{L}^c(X_G, Y_L)$ and let $M \in \mathcal{L}_G$ be the semisimple HC-vertex of D_G. We show that $M \in \mathcal{L}_{G,L}$. By general assumption 2.3 χ_G occurs as constituent of $\mathrm{R}_L^G \chi_L$. Hence $\mathrm{T}_M^G \chi_{G'}$ is a summand of

$$\mathrm{T}_M^G \mathrm{R}_L^G \chi_L = \sum_{x \in \mathcal{D}_{L,M}} \mathrm{R}_{L^x \cap M}^M \mathrm{T}_{L^x \cap M}^{L^x} \chi_{L^x}^x.$$

But $\mathrm{T}_{L^x \cap M}^{L^x} \chi_{L^x}^x = (0)$ unless $L^x \leq M$. This shows that for every composition factor S of $\bar{X}_G$ we have $\mathrm{T}_M^G S = (0)$ unless $L^x \leq M$ for some $x \in G$. As X_G and X_G^x are isomorphic for such an x, we may assume that the semisimple HC-vertex of D_G contains L.

The short exact sequence

$$0 \to \mathrm{Jac}(\bar{X}_G) \to \bar{X}_G \to D_G \to 0$$

yields a short exact sequence

$$0 \to \mathrm{T}_M^G \mathrm{Jac}(\bar{X}_G) \to \bigoplus_{y \in \mathcal{N}_{M,L}^G} \bar{X}_{M,y} \to \mathrm{T}_M^G D_G \to 0$$

using Theorem 2.8. Consequently an irreducible direct summand S of the head of $\mathrm{T}_M^G D_G$ has to be the head of one of the summands of the middle term, and thus by induction it has to be of the form $D_{M,y}$ for some $y \in \mathcal{N}_{M,L}^G$. By Lemma 1.15, $D_{M,y}$ is cuspidal and D_G is in the HC-series $S(G/M, D_{M,y})_k$. But $D_{M,y} = (D_{M^{y^{-1}}})^y$ by 2.10. Thus replacing M by $M^{y^{-1}}$ if necessary we may assume $y = 1$. By Theorem 2.16 the unique composition factor of $\bar{X}_G$ in the series $S(G/M, D_M)$ is $D_{G,M}$. Theorem 2.27 part (iii) tells us now that M has to be maximal with $M \in \mathcal{L}^c(X_G, Y_L)$. □

Recall our definitions in 1.2. We apply Theorem 2.27 in the special situation $M = L$:

Corollary 2.34 *Let $G' \in \mathcal{L}_{G,L}$. Then*

$$\rho_{G'} = \rho_{G',L} : X_{G'} \to \mathrm{R}_L^{G'} X_L$$

is injective and its image $X_\rho = X_{\rho,G'} = \rho_{G'}(X_{G'})$ is a sublattice of $X'_{G'} \leq \mathrm{R}_L^{G'} X_L$ satisfying $KX_\rho = KX'_{G'}$ and $X'_{G'} = \sqrt{\rho(X_{G'})}$. Morover every homomorphism from $Y_{G'}$ into $\mathrm{R}_L^{G'} X_L$ factors through a multiple of ρ and is the combined image

$$X_\rho = \tau_{Y_{G'}}(\mathrm{R}_L^{G'} X_L) = \tau_{Y_{G'}}(X'_{G'})$$

(which means X_ρ is the combined image of $Y_{G'}$ in $\mathrm{R}_L^{G'} X_L$ and $X_{G'}$ respectively). Finally the composition factors of $\ker \bar{\rho}$ and the cokernel $V = \mathrm{coker}\bar{\rho} = \bar{X}'_{G'}/\bar{X}_\rho$ coincide (multiplicities counted). In particular no composition factor of V has the form $D_{G',M'}$ for some $M' \in \mathcal{L}^c(X_{G'}, Y_L)$.

Proof By Theorem 2.13 and Frobenius reciprocity

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}G'}(Y_{G'}, \mathrm{R}_L^{G'} X_L) &\cong \bigoplus_{x \in \mathcal{N}_{L,L}^G} \mathrm{Hom}_{\mathcal{O}L}(Y_{L,x}, X_L) \oplus \mathrm{Hom}_{\mathcal{O}L}(Z, X_L) \\ &\cong \mathrm{Hom}_{\mathcal{O}L}(Y_L, X_L) \\ &\cong \mathcal{O},\end{aligned}$$

since $X_{L,x} \not\cong X_L$ for $1 \neq x \in \mathcal{N}_{L,L}^G$ by Theorem 2.8, $\pi_{(L,\chi_L)}(Y_L) = X_L$ and since X_L is cuspidal, by our basic assumption 2.1. Note that $\pi_{(L,\chi_L)}(Z) = (0)$ by Theorem 2.13. Thus there is a unique homomorphism $\hat{\rho} : Y_{G'} \to R_L^{G'} X_L$ corresponding to $1 \in \mathcal{O}$. Again by Lemma 1.18

$$\mathrm{Hom}_{\mathcal{O}G'}(Y_{G'}, \mathrm{R}_L^{G'} X_L) \cong \mathrm{Hom}_{\mathcal{O}G'}(X_{G'}, \mathrm{R}_L^{G'} X_L),$$

thus every homomorphism from $Y_{G'}$ to $\mathrm{R}_L^{G'} X_L$ factors through $X_{G'}$, and we can choose $\rho : X_{G'} \to \mathrm{R}_L^{G'} X_L$ such that $\hat{\rho}$ is the composite of the natural projection of $Y_{G'}$ onto $X_{G'} = \pi^{G'}(Y_{G'})$ and ρ. Thus $\mathrm{im}\,\rho$ is the combined image of $Y_{G'}$ in $\mathrm{R}_L^{G'} X_L$ and in every submodule of $\mathrm{R}_L^{G'} X_L$ containing $\operatorname{im}\rho$. Now $1_K \otimes_{\mathcal{O}} \rho : KX_{G'} \to \mathrm{R}_L^{G'} KX_L$ has to be injective since $KX_{G'}$ is irreducible by Theorem 2.8, hence ρ is injective, too. Thus $1_K \otimes_{\mathcal{O}} \rho$ maps $KX_{G'}$ isomorphically onto $KX'_{G'} \leq K\mathrm{R}_L^{G'} X_L$. In particular, since

$$X'_{G'} = \sqrt{X'_{G'}} = KX'_{G'} \cap \mathrm{R}_L^{G'} X_L = K\rho(X_{G'}) \cap \mathrm{R}_L^{G'} X_L = \sqrt{\rho(X_{G'})}$$

we have

$$\rho(X_{G'}) \leq X'_{G'}.$$

Since $KX_{G'} = KX'_{G'}$ the composition factors of $\bar{X}_{G'}$ and $\bar{X}'_{G'}$ coincide. Now by the first isomorphism theorem

$$\bar{X}_{G'} / \ker \bar{\rho} \cong \operatorname{im} \bar{\rho} \leq \bar{X}'_{G'}.$$

Consequently $\ker \bar{\rho}$ and $\operatorname{coker}\bar{\rho}$ have the same composition factors. □

Lemma 2.35 *Let $G' \in \mathcal{L}_{G,L}$ and let S be a composition factor of $\bar{X}_{G'}$ with $S \neq D_{G',M'}$ for all $M' \in \mathcal{L}^c(X_{G'}, Y_L)$. Let $M \in \mathcal{L}_{G,L}$. Then*

$$\operatorname{Hom}_{k\tilde{G}}(\mathrm{R}_M^{\tilde{G}} \bar{Y}_M, \mathrm{R}_{G'}^{\tilde{G}} S) = (0).$$

Thus no composition factor of $\mathrm{R}_{G'}^{\tilde{G}} S$ is contained in the HC-series $S(\tilde{G}/M, D_M)_k$.

Proof We have

$$\begin{aligned} \operatorname{Hom}_{k\tilde{G}}(\mathrm{R}_M^{\tilde{G}} \bar{Y}_M, \mathrm{R}_{G'}^{\tilde{G}} S) &= \operatorname{Hom}_{kM}(\bar{Y}_M, \mathrm{T}_M^G \mathrm{R}_{G'}^G S) \\ &= \bigoplus_{z \in \tilde{\mathcal{D}}_{MG'} \cap \tilde{N}_{L \subseteq M}} \operatorname{Hom}_{kM}(\bar{Y}_M, \mathrm{R}^M_{M\cap G'^z} \mathrm{T}^{G'^z}_{M\cap G'^z} S^z) \\ &= \bigoplus_{z \in \tilde{\mathcal{D}}_{MG'} \cap \tilde{N}_{L \subseteq M}} \operatorname{Hom}_{k(M\cap G'^z)}(\mathrm{T}^M_{M\cap G'^z} \bar{Y}_M, \mathrm{T}^{G'^z}_{M\cap G'^z} S^z), \end{aligned} \tag{2.36}$$

where $\tilde{\mathcal{D}}_{MG'}$ denotes a suitable system of double coset representatives of parabolic subgroups of $\tilde{G}$ containing G' and M as Levi complements. Now

$$\begin{aligned} &\operatorname{Hom}_{k(M\cap G'^z)}(\mathrm{T}^M_{M\cap G'^z} \bar{Y}_M, \mathrm{T}^{G'^z}_{M\cap G'^z} S^z) \\ &= \bigoplus_{x \in \mathcal{N}^M_{M\cap G'^z, L^z}} \operatorname{Hom}_{k(M\cap G'^z)}(\bar{Y}_{M\cap G'^z, x}, \mathrm{T}^{G'^z}_{M\cap G'^z} S^z) \oplus \operatorname{Hom}_{k(M\cap G'^z)}(\bar{Z}_z, \mathrm{T}^{G'^z}_{M\cap G'^z} S^z), \end{aligned} \tag{2.37}$$

where Z_z is a projective $\mathcal{O}M \cap G'^z$-module with $\pi^{M\cap G'^z}(Z_z) = (0)$ by Theorem 2.13.

Now S^z is a subfactor of $\bar{X}^z_{G'}$. By Corollary 2.11, applied to the projective restriction system $\mathcal{PR}^z(X_G, Y_L) = \mathcal{PR}(X^z_G, Y^z_L)$, we have

$$\mathrm{T}^{G'^z}_{M\cap G'^z} X^z_{G'} = \bigoplus_{y \in \mathcal{N}^{G'^z}_{M\cap G'^z, L^z}} X^{(z)}_{M\cap G'^z, y},$$

with $X^{(z)}_{M\cap G'^z,y} = \pi_{(L^{zy},\chi^{zy})}(\mathrm{T}^{G'^z}_{M\cap G'^z} X^z_{G'})$ (see Notation 2.6). Similarly we denote the projective cover of $X^{(z)}_{M\cap G'^z,y}$ by $Y^{(z)}_{M\cap G'^z,y}$ and we write $D^{(z)}_{M\cap G'^z,M'^x}$ for the composition factors of $\mathrm{T}^{G'^z}_{M\cap G'^z} X^z_{G'}$ lying in series $S((M\cap G'^z)/M'^x, D_{M'^x})$ for $M'^x \in \mathcal{L}^c(X^z_{G'}, Y^z_L)$.

From Theorem 2.13 and Corollary 1.24 follows

$$\mathrm{T}^{G'^z}_{M\cap G'^z} \bar{X}^z_{G'} = \mathrm{T}^{G'^z}_{M\cap G'^z} \pi^{M\cap G'^z} \bar{Y}^z_{G'} = \pi^{G'^z} \mathrm{T}^{G'^z}_{M\cap G'^z} \bar{Y}^z_{G'},$$

and hence by Lemma 1.19

$$\begin{aligned}
\mathrm{Hom}_{k(M\cap G'^z)}(\bar{Z}_z, \mathrm{T}^{G'^z}_{M\cap G'^z} S^z) &\leq \mathrm{Hom}_{k(M\cap G'^z)}(\bar{Z}_z, \mathrm{T}^{G'^z}_{M\cap G'^z} \bar{X}^z_{G'}) \\
&\cong \mathrm{Hom}_{k(M\cap G'^z)}(\bar{Z}_z, \pi^{G'^z} \mathrm{T}^{G'^z}_{M\cap G'^z} \bar{Y}^z_{G'}) \\
&\cong \mathrm{Hom}_{k(M\cap G'^z)}(\pi^{G'^z} \bar{Z}_z, \mathrm{T}^{G'^z}_{M\cap G'^z} \bar{X}^z_{G'}) \\
&\cong \mathrm{Hom}_{k(M\cap G'^z)}((0), \mathrm{T}^{G'^z}_{M\cap G'^z} \bar{X}^z_{G'}) \\
&= (0).
\end{aligned}$$

Now suppose that

$$\mathrm{Hom}_{k(M\cap G'^z)}(\bar{Y}_{M\cap G'^z,x}, \mathrm{T}^{G'^z}_{M\cap G'^z} S^z) \leq \bigoplus_{y\in \mathcal{N}^{G'^z}_{M\cap G'^z, L^z}} \mathrm{Hom}_{k(M\cap G'^z)}(\bar{Y}_{M\cap G'^z,x}, \bar{X}^{(z)}_{M\cap G'^z,y}) \neq (0),$$

say $\mathrm{Hom}_{k(M\cap G'^z)}(\bar{Y}_{M\cap G'^z,x}, \bar{X}^{(z)}_{M\cap G'^z,y_0}) \neq (0)$. Then using again Theorem 2.13 we see that $\bar{Y}_{M\cap G'^z,x}$ is the projective cover of $\bar{X}^{(z)}_{M\cap G'^z,y_0}$. That is, we have $\bar{Y}_{M\cap G'^z,x} \cong \bar{Y}^{(z)}_{M\cap G'^z,y_0}$, hence its head is one of the modules $D^{(z)}_{M\cap G'^z,M'^x}$ for some $M'^x \in \mathcal{L}^c_{G'^z,L^z}$ by Corollary 2.33. Now Theorem 2.25 implies $\mathrm{Hom}_{k(M\cap G'^z)}(\bar{Y}_{M\cap G'^z,x}, \mathrm{T}^{G'^z}_{M\cap G'^z} S^z) = (0)$ and the lemma follows. □

In the next result we extend Corollary 2.34 to HC-induced lattices. We define

$$\mathbf{Y} = \mathbf{Y}^{\tilde{G}} = \bigoplus_{G'\in\mathcal{L}_{G,L}} \mathrm{R}^{\tilde{G}}_{G'} Y_{G'}. \tag{2.38}$$

Corollary 2.39 *Let $G' \in \mathcal{L}_{G,L}$ and let $X_\rho \leq X'_{G'} \leq \mathrm{R}^{G'}_L$ be defined as in Corollary 2.34. Then*

$$\mathrm{R}^{\tilde{G}}_{G'} X_\rho = \tau_{\mathbf{Y}}(\mathrm{R}^{\tilde{G}}_{G'} X'_{G'}).$$

Proof By 2.34 X_ρ is image of $Y_{G'}$ hence $\mathrm{R}^{\tilde{G}}_{G'} X_\rho$ is epimorphic image of $\mathrm{R}^{\tilde{G}}_{G'} Y_{G'}$, which is a direct summand of $\mathbf{Y}$. We conclude that

$$\mathrm{R}^{\tilde{G}}_{G'} X_\rho \leq \tau_{\mathbf{Y}}(\mathrm{R}^{\tilde{G}}_{G'} X'_{G'}).$$

Let S be a composition factor of the cokernel of $\bar{\rho}$. Then, by Lemma 2.34 S is not of the form $D_{G',M'}$ for some $M' \in \mathcal{L}^c(X_{G'}, Y_L)$. Lemma 2.35 tells us

$$\mathrm{Hom}_{k\tilde{G}}(\mathrm{R}^{\tilde{G}}_M \bar{Y}_M, \mathrm{R}^{\tilde{G}}_{G'} S) = (0).$$

Thus the image of every homorphism from $\bar{\mathbf{Y}}$ to $\mathrm{R}_{G'}^{\tilde{G}}\bar{X}_{G'}$ has to be contained in $\operatorname{im}\bar{\rho}$. We lift this to $\mathcal{O}$ to get the desired result. □

Lemma 2.40 *Suppose that every composition factor of* $\mathrm{R}_L^G\bar{X}_L$ *is isomorphic to some* $D_{G,M}$ *where* $M \in \mathcal{L}^c(X_G, Y_L)$. *Then we have*

(i) *The* kL*-module* $\bar{X}_L$ *is irreducible, that is* $\bar{X}_L = D_L$.

(ii) *Let* $G' \in \mathcal{L}_{G,L}$. *Then every composition factor of* $R_L^{G'}\bar{X}_L$ *is of the form* $D_{G',M}$ *for some* $\mathcal{L}^c(X_{G'}, Y_L) = \{M \in \mathcal{L}^c(X_G, Y_L) \mid M \leq G'\}$.

(iii) *There is an isomorphism between the poset of local submodules of* $\bar{X}_{G'}$ *and the poset of subgroups* $\mathcal{L}^c(X_{G'}, Y_L)$ *of* G'.

(iv) $R_{G'}^{\tilde{G}}X_{G'}$ *is a pure sublattice of* $R_L^{\tilde{G}}\chi_L$ *isomorphic to* $R_{G'}^{\tilde{G}}X'_{G'}$ *for all* $G' \in \mathcal{L}_{G,L}$.

Proof Lemma 2.35 implies immediately part (i) and Theorem 2.27 part (iii) whence we have established part (ii) observing that all composition factors of $\bar{X}$ are composition factors of $R_L^G\bar{X}_L$, since KX_G is a constituent of $R_L^{G'}X_L$ by Definition 2.3.

To show part (ii) let D be a composition factor of $R_L^{G'}\bar{X}_L$ and let $D \in S(G'/M', E)_k$, where E is an irreducible cuspidal kM'-module and $M' \in \mathcal{L}_{G'}$. Since D is a composition factor of $R_L^{G'}\bar{X}_L$ and $X \in \mathfrak{Cat}_{\mathcal{O}L}$ is cuspidal, we may choose M' such that $L \leq M'$, that is $M' \in \mathcal{L}_{G',L}$. Now $R_{G'}^G(D)$ is a subfactor of $R_L^G(\bar{X}_L)$ and, by Lemma 1.14, every composition factor of the head of $R_{G'}^G(D)$ is in $S(G/M', E)_k$. Our general assumptions imply now that $S(G/M', E)_k = S(G/M, D_M)_k$ for some $M \in \mathcal{L}^c(X_G, Y_L)$. Thus there exists $x \in N$ such that $M' = M^x$ and $D_M^x = E$. Since $M' \in \mathcal{L}_{G,L}$ we have $M' \in \mathcal{L}^c(X_G, Y_L)$ and $E = D_{M'}$. But then $D \in S(G'/M', D_{M'})$ as desired.

It remains to show (iv): In Theorem 2.27 we constructed a homomorphism $0 \neq \rho_{G',L} : X_{G'} \to R_L^{G'}X_L$, which is injective since KG' is an irreducible KG'-module. Thus $X_{G'} \subset R_L^{G'}X_L$. But part *ii*) above and part *i*) of Theorem 2.27 imply that $\bar{\rho}_{G',L} := 1_k \otimes_{\mathcal{O}} \rho_{G',L} : \bar{X}_{G'} \to R_L^{G'}\bar{X}_L$ is injective, too, hence $X_{G'}$ is a pure sublattice of $R_L^{G'}X_L$. By construction $X'_{G'}$ is a pure sublattice of $R_L^{G'}X_L$ affording the same character $\chi_{G'}$. Since the multiplicity of $\chi_{G'}$ in $R_L^{G'}\chi_L$ is one, $X'_{G'}$ has to be the image of $\rho_{G',L}$. Since HC-induction preserves purity and isomorphisms part (iv) follows □

We point out that the assumption in Corollary 2.40 is frequently satisfied, as for example in the application for $GL_n(q)$ and we shall see in the following section how the conclusion of Corollary 2.40 can be used to get information on decomposition numbers of $\tilde{G}$.

3 Hecke- and Schur algebras

We continue with the set up of the previous section. So let $\mathcal{PR} = \mathcal{PR}(X_G, Y_L)$ be a projective restriction system, consisting of the data $\{X_M, Y_M, X'_M | M \in \mathcal{L}_{G,L}\}$. We define

$Q = Q_L = \mathrm{R}_L^{\tilde{G}} Y_L$ and $\beta^{\tilde{G}} = \beta = \mathrm{R}_L^{\tilde{G}}(\psi)$, where $\psi : Y_L \to \pi^L(Y_L) = X_L$ is the natural projection.

Theorem 3.1 *The epimorphism $\beta : Q \to \mathrm{R}_L^{\tilde{G}} X_L$ satisfies Hypothesis* 1.1. *Thus in particular the endomorphism ring of $\mathrm{R}_L^{\tilde{G}} \bar{X}_L$ is liftable and the decomposition matrix of the endomorphism ring $\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathrm{R}_L^{\tilde{G}} X_L)$ is part of the ℓ-modular decomposition matrix of $\tilde{G}$.*

Proof By Corollary 1.24 and by 2.1

$$\pi^{\tilde{G}}(Q) = \pi^{\tilde{G}}(\mathrm{R}_L^{\tilde{G}} Y_L) = \mathrm{R}_L^{\tilde{G}} \pi^L(Y_L) = \mathrm{R}_L^{\tilde{G}} X_L,$$

hence the theorem follows from Result 1.25. □

Lemma 1.8 implies now immediately:

Corollary 3.2 *The $\mathcal{O}\tilde{G}$-lattice $\mathrm{R}_L^{G'} X_L$ is Q-torsionless.*

Notation 3.3 We call $\mathcal{H}_R^{\tilde{G}} = \mathcal{H}_R^{\tilde{G}}(\mathcal{PR}) = \mathrm{End}_{R\tilde{G}}(R \otimes_{\mathcal{O}} \mathrm{R}_L^{\tilde{G}} X_L)$ the *Hecke algebra* over R *associated* with $\mathcal{PR} = \mathcal{PR}(X_G, Y_L)$ on level $\tilde{G}$. If no ambiguities arise we omit super- and subscripts, for instance $\mathcal{H}_R$ is until further notice $\mathcal{H}_R^{\tilde{G}}$. Observe that Theorem 3.1 implies that $\mathcal{H}_R = R\mathcal{H}_{\mathcal{O}} = R \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}$. According to the general theory of quotients of Hom-functors as outlined in the first section, we have associated with $\mathcal{H}_R$ quotients of Hom-functors H_R given by

$$H_R : \mathfrak{Mod}_{R\tilde{G}} \to \mathfrak{Mod}_{\mathcal{H}_R} : V \to \mathrm{Hom}_{R\tilde{G}}(Q, V)/\mathrm{Hom}_{R\tilde{G}}(Q, V) J_\beta,$$

where J_β is the ideal of $\mathrm{End}_{R\tilde{G}}(Q)$ consisting of those endomorphisms of Q, whose image is contained in the kernel of β.

Let $G' \in \mathcal{L}_{G,L}$. We have seen in Lemma 2.34 that the irreducible character $\chi_{G'}$ of $X_{G'}$ occurs exactly with multiplicity one in $\mathrm{R}_L^{G'} \chi_L$. By Fittings Lemma we find a one dimensional $\mathcal{H}_K^{G'}$-module generated by $\sigma = \sigma_{G'}$ such that $K\sigma \mathrm{R}_L^{G'} X_L = K X_{G'}$. We choose $\sigma \in \mathcal{H}_{\mathcal{O}}^{G'}$ such that $\sigma \mathcal{H}_{\mathcal{O}}^{G'}$ is pure in $\mathcal{H}_{\mathcal{O}}^{G'}$ and is a generator of this one dimensional $\mathcal{O}$-space. Then $\sigma \mathrm{R}_L^{G'} X_L$ is an $\mathcal{O}$-lattice in $K X_{G'}$, but in general $\sigma \mathrm{R}_L^{G'} X_L \ncong X_{G'}$ and $\sigma \mathrm{R}_L^{G'} X_L \ncong X'_{G'} \leq \mathrm{R}_L^{G'} X_L$. However $X'_{G'} = \iota_{(G', \chi_{G'})}(\mathrm{R}_L^{G'} X_L)$ is pure in $\mathrm{R}_L^{G'} X_L$ by construction and affords $\chi_{G'}$. This implies:

Lemma 3.4 *Let $U \leq \mathrm{R}_L^{G'} X_L$ with $KU = K X'_{G'}$. Then*

$$U \leq X'_{G'} = \sqrt{U}.$$

In particular

$$\sigma \mathrm{R}_L^{G'} X_L \leq X'_{G'} \quad \textit{and} \quad \sqrt{\sigma \mathrm{R}_L^{G'} X_L} = X'_{G'}.$$

Proof We have

$$\sqrt{U} = KU \cap \mathrm{R}_L^G X_L = KX'_G \cap \mathrm{R}_L^G X_L = X'_G.$$

$\square$

The next result relates $\mathcal{H}_{\mathcal{O}}^{G'}$ and $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$:

Theorem 3.5 *Let $G' \in \mathcal{L}_{\tilde{G},L}$. Then the following holds for $R \in \{K, \mathcal{O}, k\}$:*

(i) *$\mathcal{H}_R^{G'}$ is a subalgebra of $\mathcal{H}_R^{\tilde{G}}$ and $\mathcal{H}_R^{\tilde{G}}$ is free as left $\mathcal{H}_R^{G'}$-module.*
(ii) *If I is a right ideal of $\mathcal{H}_{\mathcal{O}}^{G'}$ and $Z = I\mathrm{R}_L^{G'} X_L$ then*

$$\mathrm{R}_{G'}^{\tilde{G}} Z = I\mathcal{H}_{\mathcal{O}}^{\tilde{G}} \mathrm{R}_L^{\tilde{G}} X_L = I\mathrm{R}_L^{\tilde{G}} X_L,$$

where $I\mathcal{H}_{\mathcal{O}}^{\tilde{G}} \cong I \otimes_{\mathcal{H}_{\mathcal{O}}^{G'}} \mathcal{H}_{\mathcal{O}}^{\tilde{G}}$. In particular, if I is pure in $\mathcal{H}_{\mathcal{O}}^{G'}$ then $I\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ is pure in $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$.

In particular, the lattice $\sigma_{G'}\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ is a pure right ideal of $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$.

Proof By Frobenius reciprocity

$$\mathcal{H}_{\mathcal{O}}^{\tilde{G}} = \mathrm{Hom}_{\mathcal{O}\tilde{G}}(\mathrm{R}_L^{\tilde{G}} X_L, \mathrm{R}_L^{\tilde{G}} X_L) = \mathrm{Hom}_{\mathcal{O}G'}(\mathrm{R}_L^{G'} X_L, \mathrm{T}_{G'}^{\tilde{G}} \mathrm{R}_L^{\tilde{G}} X_L).$$

Note that maps in the right hand side Hom-set correspond to maps in the left Hom-set simply by extending the map in the natural way to $\mathrm{R}_L^{\tilde{G}} X_L = \mathrm{R}_L^{G'} X_L \mathcal{O}\tilde{G}$, where $\mathrm{R}_L^{G'} X_L$ is considered as module for a parabolic subgroup whose Levi complement is G'. We choose a set $\tilde{\mathcal{D}}_{L,G'}$ of double coset representatives of parabolic subgroups of $\tilde{G}$ having L and G' as Levi complements. Now by Mackey decomposition (see 2.7)

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}G'}(\mathrm{R}_L^{G'} X_L, \mathrm{T}_{G'}^{\tilde{G}} \mathrm{R}_L^{\tilde{G}} X_L) &\cong \bigoplus_{x \in \tilde{\mathcal{D}}_{L,G'}} \mathrm{Hom}_{\mathcal{O}G'}(\mathrm{R}_L^{G'} X_L, \mathrm{R}_{G' \cap L^x}^{G'} \mathrm{T}_{G' \cap L^x}^{L^x} X_L^x) \\ &\cong \bigoplus_{x \in \tilde{\mathcal{D}}_{L,G'} \cap N_{L \subseteq G'}} \mathrm{Hom}_{\mathcal{O}G'}(\mathrm{R}_L^{G'} X_L, \mathrm{R}_{L^x}^{G'} X_L^x), \end{aligned} \tag{3.6}$$

since X_L is cuspidal. But for $x = 1$ we get $\mathcal{H}_{\mathcal{O}}^{G'}{}' = \mathrm{Hom}_{\mathcal{O}G'}(\mathrm{R}_L^{G'} X_L, \mathrm{R}_L^{G'} X_L)$ and for $1 \neq x \in \tilde{\mathcal{D}}_{L,G'}$, $L^x \leq G'$, $\mathrm{Hom}_{\mathcal{O}G'}(\mathrm{R}_L^{G'} X_L, \mathrm{R}_{L^x}^{G'} X_L^x)$ is either (0), if $L^x \neq_{G'} L$, or $L^{xy} = L$ for a $y \in G'$ but $X_L^{xy} \ncong X_L$. Or it is as left $\mathcal{H}_{\mathcal{O}}^{G'}$-module isomorphic to the regular module. A basis of this is then given for instance by choosing one isomorphism from $\mathrm{R}_L^{G'} X_L$ to $\mathrm{R}_{L^x}^{G'} X_L^x$ induced by conjugation certain $y \in G'$. For $R = k, K$ we argue analogously. Part ii) follows now easily. $\square$

Notation 3.7 Recall the definition of $\rho_{G'} : X_{G'} \to \mathrm{R}_L^{G'} X_L$ in Corollary 2.34. The following $\mathcal{O}\tilde{G}$-lattices will be considered:

$$\begin{aligned}
\mathbf{X}' &= \mathbf{X}'^{\tilde{G}} = \bigoplus_{G' \in \mathcal{L}_{G,L}} \mathrm{R}_{G'}^{\tilde{G}} X'_{G'} \\
\mathbf{X} &= \mathbf{X}^{\tilde{G}} = \bigoplus_{G' \in \mathcal{L}_{G,L}} \mathrm{R}_{G'}^{\tilde{G}} X_{G'} \\
\mathbf{X}_h &= \mathbf{X}_h^{\tilde{G}} = \bigoplus_{G' \in \mathcal{L}_{G,L}} \sigma_{G'} \mathrm{R}_{G'}^{\tilde{G}} X_L \\
\mathbf{Y} &= \mathbf{Y}^{\tilde{G}} = \bigoplus_{G' \in \mathcal{L}_{G,L}} Q_{G'},
\end{aligned}$$

setting $Q_{G'}^{\tilde{G}} = Q_{G'} = \mathrm{R}_{G'}^{\tilde{G}} Y_{G'}$. Define $\boldsymbol{\rho} : \mathbf{X} \to \mathbf{X}'$ to be the homomorphism obtained by summing up the maps $\mathrm{R}_{G'}^{\tilde{G}} \rho_{G'}$, for $G' \in \mathcal{L}_{G,L}$. As in Corollary 2.34 we denote the image of $\rho = \rho_{G'}$ in $X'_{G'}$ by $X_\rho = X_{\rho_{G'}}$. Note that $kX_\rho = \bar{X}_\rho$ is not the image of $\bar{\rho} = 1_k \otimes_{\mathcal{O}} \rho$ in general, but the latter is always an epimorhic image of the former.

Here is our main result:

Theorem 3.8 *Let the $\mathcal{O}\tilde{G}$-lattices $\mathbf{X} = \mathbf{X}^{\tilde{G}}$ and $\mathbf{Y} = \mathbf{Y}^{\tilde{G}}$ be defined as in* 3.7. *Then*

$$\pi^{\tilde{G}}(\mathbf{Y}) = \mathbf{X}.$$

Let $\boldsymbol{\beta} : \mathbf{Y} \to \mathbf{X}$ denote the corresponding epimorphism. Then $\boldsymbol{\beta}$ satisfies Hypothesis 1.1 *and the requirements of Theorem* 1.7. *In particular, the decomposition matrix of* $\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X})$ *is part of the ℓ-modular decomposition matrix of $\tilde{G}$.*

Proof Corollary 2.14 implies

$$\pi^{\tilde{G}}(Q_M) = \pi^{\tilde{G}}(\mathrm{R}_M^{\tilde{G}} Y_M) = \mathrm{R}_M^{\tilde{G}} X_M$$

for every $M \in \mathcal{L}_{G,L}$. Since $\pi^{\tilde{G}}$ obviously preserves direct sums the theorem follows using Corollary 1.25. □

Lemma 3.4 in conjunction with Theorem 1.10 imply that $\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X}_h) = \mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X}')$ (see 3.12 below). Now Lemma 2.40 implies immediately:

Theorem 3.9 *Suppose that every composition factor of* $\mathrm{R}_L^G \bar{X}_L$ *is isomorphic to some $D_{G,M}$ where $M \in \mathcal{L}^c(X_G, Y_L)$. Then $\mathbf{X} \cong \mathbf{X}_{\boldsymbol{\rho}} = \mathbf{X}'$.*

So under the assumption of the theorem we can immediately compute the decoposition of irreducible lattices occurring in $\mathbf{X}$ in terms of the Hecke algebra $\mathcal{H}_{\mathcal{O}}$.

Obviously we would now like to know $\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X})$ in general. We shall use the following result to show, that we can relate this endomorphism ring to the endomorphism rings of the other lattices defined in 3.7.

Theorem 3.10 *Keep the notation introduced above. Let $G' \in \mathcal{L}_{G,L}$.*

(i) *We have $\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L \leq \mathrm{R}_{G'}^{\tilde{G}} X_\rho \leq \mathrm{R}_{G'}^{\tilde{G}} X'_{G'}$, and therefore*

$$\mathbf{X}_h \leq \mathbf{X}_{\boldsymbol{\rho}} \leq \mathbf{X}',$$

where $\mathbf{X}_{\boldsymbol{\rho}} = \mathbf{X}_{\boldsymbol{\rho}}^{\tilde{G}} = \boldsymbol{\rho}(\mathbf{X})$.

(ii) $\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L = \tau_{Q_L}(\mathrm{R}_{G'}^{\tilde{G}} X_\rho) = \tau_{Q_L}(\mathrm{R}_{G'}^{\tilde{G}} X'_{G'})$ *and* $\tau_{Q_{G'}}(\mathrm{R}_{G'}^{\tilde{G}} X_{G'}) = \mathrm{R}_{G'}^{\tilde{G}} X_\rho$, *hence*

$$\tau_{Q_L}(\mathbf{X}') = \mathbf{X}_h = \tau_{Q_L}(\mathbf{X}_{\boldsymbol{\rho}}) \quad \textit{and} \quad \tau_{\mathbf{Y}}(\mathbf{X}') = \mathbf{X}_{\boldsymbol{\rho}}.$$

Proof Note that $K\sigma_{G'} \mathrm{R}_L^{G'} X_L = KX_\rho = KX'_{G'}$. Thus Lemma 3.4 implies

$$\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L \leq \mathrm{R}_{G'}^{\tilde{G}} X_{G'} \quad \text{and} \quad \mathrm{R}_{G'}^{\tilde{G}} X_\rho \leq \mathrm{R}_{G'}^{\tilde{G}} X'_{G'},$$

since HC-induction preserves injections.

Corollary 2.39 implies immediately

$$\tau_{Q_M}(\mathrm{R}_{G'}^{\tilde{G}} X'_{G'}) \leq \mathrm{R}_{G'}^{\tilde{G}} X_\rho = \tau_{\mathbf{Y}}(\mathrm{R}_{G'}^{\tilde{G}} X'_{G'}) \tag{3.11}$$

for every $M \in \mathcal{L}_{G,L}$, since Q_M is a direct summand of $\mathbf{Y}$. Taking $M = L$ we obtain

$$\tau_{Q_L}(\mathrm{R}_{G'}^{\tilde{G}} X'_{G'}) \leq \mathrm{R}_{G'}^{\tilde{G}} X_\rho.$$

Now $\sigma_{G'} \mathrm{R}_L^{G'} X_L$ is the image of the endomorphism σ of $\mathrm{R}_L^{G'} X_L$ and hence the image of the surjective composite map

$$\mathrm{R}_L^{G'} Y_L \to \mathrm{R}_L^{G'} X_L \xrightarrow{\sigma_{G'}} \sigma_{G'} \mathrm{R}_L^{G'} X_L,$$

hence $\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L$ is the image of the composite map

$$Q_L = \mathrm{R}_L^{\tilde{G}} Y_L \to \mathrm{R}_L^{\tilde{G}} X_L \xrightarrow{\hat{\sigma}} \sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L$$

setting $\hat{\sigma} = \mathrm{R}_{G'}^{\tilde{G}}(\sigma_{G'})$. We conclude that

$$\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L \leq \tau_{Q_L}(\mathrm{R}_{G'}^{\tilde{G}} X'_{G'}) \leq \mathrm{R}_{G'}^{\tilde{G}} X_\rho$$

and

$$\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L = \tau_Q(\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L).$$

Summing up we get part i). By Theorem 3.5 $\sigma_{G'} \mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ is pure in $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$, hence by [6, 4.13] and Lemma 3.4 we have

$$\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L = \tau_{Q_L}(\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L) = \tau_{Q_L}(\sqrt{\sigma_{G'} \mathrm{R}_L^{\tilde{G}} X_L}) = \tau_{Q_L}(\mathrm{R}_{G'}^{\tilde{G}} X'_{G'}).$$

Summing up we get

$$\tau_{Q_L}(\mathbf{X}') = \mathbf{X}_h = \tau_{Q_L}(\mathbf{X}_{\boldsymbol{\rho}}).$$

Applying formula 3.11 for $M = G'$ we obtain

$$\tau_{Q_{G'}}(\mathrm{R}_{G'}^{\tilde{G}} X_{G'}) \leq \mathrm{R}_{G'}^{\tilde{G}} X_\rho$$

but obviously we have equality here, since $\mathrm{R}_{G'}^{\tilde{G}} X_\rho$ is epimorphic image of $Q_{G'} = \mathrm{R}_{G'}^{\tilde{G}} Y_{G'}$. Summing up we get

$$\tau_{\mathbf{Y}}(\mathbf{X}') = \mathbf{X}_{\boldsymbol{\rho}}.$$

□

The additional hypothesis on $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ (see 1.9) in the following result is satisfied in all our applications:

Theorem 3.12 *Suppose the Hecke algebra $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ is integrally quasi Frobenius. Let* $\mathrm{T} := \mathrm{T}^{\tilde{G}}(\mathcal{PR})$ *be the $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$-module*

$$T = \bigoplus_{G' \in \mathcal{L}_{G,L}} \sigma_{G'} \mathcal{H}_{\mathcal{O}}^{\tilde{G}}.$$

(i) *Let the functor $H_{\mathcal{O}}^{\tilde{G}} = H_{\mathcal{O}}$ be defined as in* 3.3. *Then*

$$H_{\mathcal{O}}(\mathbf{X}_h) = H_{\mathcal{O}}(\mathbf{X}) = H_{\mathcal{O}}(\mathbf{X}').$$

(ii) *Restricting endomorphisms to sublattices induces isomorphisms*

$$\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X}_h) \cong \mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X}) \cong \mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X}').$$

(iii) *For every lattice $\mathbf{Z} = \mathbf{X}_h, \mathbf{X}, \mathbf{X}'$ the functor $H_{\mathcal{O}}$ induces an isomorphism*

$$H_{\mathcal{O}} : \mathrm{End}_{\mathcal{H}_{\mathcal{O}}^{\tilde{G}}}(\mathrm{T}) \to \mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{Z}).$$

Proof We may identify $\mathbf{X}$ with its image $\mathbf{X}_{\boldsymbol{\rho}}$ under the inejection $\boldsymbol{\rho}$. We apply Theorem 3.10. For any $\mathcal{O}\tilde{G}$-lattice V we have always $H_{\mathcal{O}}(V) = \mathcal{H}_{\mathcal{O}}(\tau_{Q_L}(V))$ by [6, 2.17], hence i) follows.

By Lemma 1.3 the endomorphisms of $\mathbf{X}'$ map the subspaces $\tau_{Q_L}\mathbf{X}' = \mathbf{X}_h$ and $\tau_{\mathbf{Y}}(\mathbf{X}') = \mathbf{X} = \mathbf{X}_{\boldsymbol{\rho}}$ into itself, restriction defines so an homomorphism from $\mathrm{End}_{\mathcal{O}\tilde{G}}\mathbf{X}'$ to $\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X}_h)$ and to $\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X})$ repectively. Similarly we have an homomorphism from $\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X})$ to $\mathrm{End}_{\mathcal{O}\tilde{G}}(\mathbf{X}_h)$. By Theorem 1.10 part ii) restricting endomorphisms from $\mathbf{X}'$ to $\mathbf{X}_h$ is an isomorphism between the corresponding endomorphism rings, therefore from $\mathbf{X}'$ to $\mathbf{X}$ as well. This shows part ii). Part iii) follows now from the cooresponding part of 1.10. □

In the following we call the $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$-lattice T the *parabolic tensor space* of $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$. As usual $k \otimes_{\mathcal{O}} \mathrm{T} = \bar{\mathrm{T}}$. The endomorphism ring of T is denoted by $\mathcal{S}_{\mathcal{O}} = \mathcal{S}_{\mathcal{O}}^{\tilde{G}}(\mathcal{PR})$ and is called the *parabolic q-Schur algebra* of $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$. The reason for these names is the following: We shall apply these results to finite reductive groups. Here $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ is always a Hecke algebra associated with some reflection group W extended possibly by an abelian group. The one dimensional representation $\mathcal{O}\sigma_{G'}$ of $\mathcal{H}_{\mathcal{O}}^{G'}$ is a representation analogous to the alternating representation of symmetric groups. The space T is then isomorphic to the sum of modules analogous to the permutation representation of W on its parabolic subgroups (respectively to the sum over its parabolic subgroups corresponding to the Levi subgroups $G' \in \mathcal{L}_{G,L}$), which is again for symmetric groups W a space which is closely connected to tensor space. In general, endomorphism rings of such tensor spaces are now called q-Schur algebras. In type B for instance q-Schur algebras have been defined based not only on parabolic subgroups but on the larger set of reflection subgroups, ([15] and [16]). With this notation we have now the following main result for the projective restriction system $\mathcal{PR}$ using Theorems 1.7 and 3.12:

Corollary 3.13 *Let $\mathcal{PR}$ be a projective restriction system such that the Hecke alhgebra $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ is integrally Frobenius. Then the decomposition matrix of the associated parabolic q-Schur algebra $\mathcal{S}_{\mathcal{O}}^{\tilde{G}}(\mathcal{PR})$ is part of the ℓ-modular decomposition matrix of G.*

Remark 3.14 *We remark that in general the Hecke algebras $\mathcal{H}_R^{G'}(\mathcal{PR})$ and $\mathcal{H}_R^{G''}(\mathcal{PR})$ for $G', G'' \in \mathcal{L}_{G,L}$ can be the same. This happens since $\mathcal{H}_R^{\tilde{G}}(\mathcal{PR})$ is not a Hecke algebra defined over the Weyl group $\tilde{W}$ of $\tilde{G}$ but over some subquotient, namely over the group $W(L, X_L)$ which is the stabilizer of X_L in $\mathcal{N}_{\tilde{N}}(L)/L \cong \mathcal{N}_{\tilde{W}}(W_L)/W_L$, where $W_L := (\tilde{N} \cap L)/T$ is the Weyl group of L. Note that in this case the $\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$-modules $\sigma_{G'}\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ and $\sigma_{G''}\mathcal{H}_{\mathcal{O}}^{\tilde{G}}$ are isomorphic. Thus the resulting q-Schur algebras are Morita equivalent, and we could remove one of the corresponding summands in our summation*

$$T_{\mathcal{O}} = \bigoplus_{M \in \mathcal{L}_{G,L}} \sigma_M \mathcal{H}_{\mathcal{O}}^{\tilde{G}}.$$

We have seen that part of the ℓ-modular decomposition matrix of $\tilde{G}$ can be calculated by computing decomposition numbers of parabolic q-Schur algebras, which in turn are endomorphism rings of q-tensor space of Hecke algebras, provided the latter are integrally Frobenius. In the next section we shall construct projective restriction systems which satisfy this condition.

4 Finite groups of Lie-Type

4.1 Our main applications of the theory developed in the preceding sections are on finite groups of Lie type. For any such group, say G, bold face letter denotes its underlying algebraic group, that is, for example $\mathbf{G}$ will denote the underlying algebraic group for which G is the set of fixed points of some Frobenius morphism (the latter will always be assumed to be known). There exist finite groups of Lie type which can be defined via various non-isomorphic algebraic groups. However, which one is considered will always follow from the context. For finite groups of Lie type, the theory of Deligne and Lusztig plays an important role. There, the dual groups of finite groups of Lie type and the dual groups of connected algebraic groups are considered. We denote the dual group of some finite group of Lie type or some connected algebraic group with the same letter, added upper indices '*'. For $L \in \mathcal{L}_G$ we view L^* as Levi subgroup of G^* in the usual way, after choosing an isomorphism between the root data of G and G^* (see [2, 4.2]). We assume in the following that we have chosen once and for all such an isomorphism for the actually considered group G. Then the identification of L^* with a Levi subgroup of G^* becomes canonical.

The algebraic groups we consider are defined over algebraic closures of finite fields. Therefore let $\mathbb{F}_q$ denote the field of q elements for some power q of p and let $\bar{\mathbb{F}}_q$ be its algebraic closure.

Finally, if s and z are elements of G, we write $s \sim_G z$ if s is conjugate to z in G.

Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbb{F}}_q$. Let F be a Frobenius automorphism of $\mathbf{G}$ and let G be the set of F-fixed points. For simplicity of notation we assume that the center of $\mathbf{G}$ is connected, although most of the following could be formulated more generally. By general theory the center of every Levi subgroup of $\mathbf{G}$ is connected too.

First we want to summerize some well known facts.

4.2 By the theory of Deligne and Lusztig, the ordinary characters of G are distributed in pairwise disjoint series $\mathcal{E}(G,z)$, called (rational) Lusztig series, where z runs through a set of representatives of the conjugacy classes of semi simple elements of G^*. For details see e.g. [2, chapter 7]. The Lusztig series are defined by the Deligne-Lusztig operator, which maps characters of generalized Levi subgroups of G to generalized characters of G. For our purpose it is enough to know that for $L \in \mathcal{L}_G$ this operator and HC-induction coincide ([2, 7.4.4]). In particular, the Lusztig series are unions of HC-series and $\mathrm{R}_L^G\mathcal{E}(L,z) \subset \mathcal{E}(G,z)$ for semisimple $z \in L^*$. We have therefore, using Frobenius reciprocity:

Lemma 4.3 *Two characters in different Lusztig series are contained in different HC-series. Moreover, if χ is contained in $\mathcal{E}(G,z)$ and ψ is a constituent of $\mathrm{T}_L^G\chi$, then ψ is in $\mathcal{E}(L,(\tilde{z}))$ for some semisimple $\tilde{z}$ which is conjugate in G^* to z.*

There exists a canonical isomorphism between the Weyl groups W of G and W^* of G^* (by virtue of the fixed isomorphism between their root data, see e.g. [2, 4.2.3]). As the torus T of G is contained in every $M \in \mathcal{L}_G$, the Weyl group W acts by conjugation on the set of Levi subgroups of G and on the set of pairs (M,χ), where $M \in \mathcal{L}_G$ and χ an irreducible character of M. We denote the stabilizer of M under this action by $\mathrm{stab}_W(M)$. On the other hand W^*, and thus W (via the canonical isomorphism), acts on the set of Levi subgroups of G^* and on the set of pairs $(M^*,(z))$, where $z \in M^*$ is semisimple and (z) denotes its conjugacy class in M^*. It follows immediately that $\mathrm{stab}_W(M)$ is the stabilizer of M^* as well; (the identification of M^* with a Levi subgroup of G^* is just defined that way). Now let $z \in M^*$. Then by the definition of the Deligne-Lusztig operator (see [2, 7.2]) we have

$$\mathcal{E}(M,z)^w = \mathcal{E}(M^w,z^w). \tag{4.4}$$

4.5 It was shown by Broué and Michel in [1, 2.2] that for an ℓ-regular semisimple element $z \in G^*$, the set

$$\mathcal{E}_\ell(G,z) = \bigcup_{t \in C_{G^*}(z)_\ell} \mathcal{E}(G,zt) \tag{4.6}$$

forms a union of ℓ-modular blocks, where $C_{G^*}(z)_\ell$ denotes the ℓ-elements of $C_{G^*}(z)$. Moreover, Geck and Hiss have shown in [21] that the characters in $\mathcal{E}(G,z)$ form a basic set of the blocks $\mathcal{E}_\ell(G,z)$ if ℓ is a good prime for G (e. g. ℓ arbitrary if G is of type A and ℓ odd if G is of type B, C or D). That is their reduction modulo ℓ forms a $\mathbb{Z}$-basis of the additive group of generalized Brauer characters in the blocks $\mathcal{E}_\ell(G,z)$. Therefore, the decomposition matrix of the characters in $\mathcal{E}(G,z)$ determine uniquely the decomposition matrix of the whole blocks $\mathcal{E}_\ell(G,z)$. Moreover, the latter can be derived from the former by decomposition of Deligne-Lusztig induced characters, as shown in [21]. An important property of G is that the Hecke algebra $\mathrm{End}_{KG}(\mathrm{R}_L^G X)$ for some cuspidal irreducible KL-module X is always untwisted by [41, 4.23].

4.7 We want to exhibit a projective restriction system $\mathcal{PR}(X_G, Y_L)$ for (L, χ_L) and some $L \in \mathcal{L}_G$. An important role for the representations of G plays a certain character, the *Gelfand-Graev* character $\Gamma = \Gamma_G$ of G. For its definition and basic properties we refer to the standard literature, e.g. [2] and we list only a few facts on Γ which will be needed later. One of its main features is that many cuspidal irreducible characters are constituents of Γ and that all irreducible constituents of Γ occur with multiplicity one in it. These constituents are called *regular* characters. Each Lusztig series contains precisely one regular character. Another important property of Γ is that its HC-restriction $\mathrm{T}_{G'}^G(\Gamma_G)$ to the Levi subgroup G' of G gives the Gelfand-Graev character $\Gamma_{G'}$ of G'. Moreover Γ is induced from a linear character of the unipotent radical of a Borel subgroup which is in particular ℓ-regular. As a consequence there is a unique projective $\mathcal{O}G$-lattice P_G affording Γ. Moreover,

$$\mathrm{T}_{G'}^G P_G = P_{G'}. \tag{4.8}$$

Let χ be an irreducible constituent of Γ_G. Then using the notation introduced in 1.16 we conclude that $\pi_\chi^G(P_G)$ is a lattice V affording the character χ. Now

$$\mathrm{Hom}_{KG}(P_G, \mathrm{R}_L^G KX_L) = \mathrm{Hom}_{KL}(\mathrm{T}_L^G\Gamma_G, KX_L) = \mathrm{Hom}_{KL}(\Gamma_L, KX_L),$$

where X_L is a lattice affording χ_L. Thus the multiplicity of χ in $\mathrm{R}_L^G\chi_L$ is one if $\chi \in \mathcal{E}(G,z)$ and χ_L is regular, and zero otherwise.

4.9 For the following result we also need the fact, due to Hiss, ([30, 4.6.1]), that for a given block $\mathcal{B}$ of G the summand of P_G belonging to $\mathcal{B}$ is indecomposable. In fact every union 4.6 of blocks of G contains exactly one such indecomposable direct summand of P_G. Let $M \in \mathcal{L}_G$ and let $\mathcal{B}$ be a union 4.6 of blocks of M containing the Lusztig series $\mathcal{E}(M,y)$, where $y \in M^*$ is semi simple. Then we denote the unique indecomposable direct summand of P_M in $\mathcal{B}$ by $Y_{M,(y)}$. Since every Lusztig series contains a constituent of P_M, the summand $Y_{M,(y)}$ is well defined, however the label y is only determined up to its ℓ-part by the indecomposable direct summand of P_M. In particular we have a bijection between direct indecomposable summands of P_M and M^*-conjugacy classes of semi simple elements of order prime to ℓ.

Now consider the character ψ of $\mathrm{T}_{G'}^G Y_{G,(y)}$ for some semisimple ℓ-regular element $y \in G^*$. Then 4.3 implies that ψ consists precisely of those summands of $\Gamma_{G'} = \mathrm{T}_{G'}^G\Gamma_G$ which are in some $\mathcal{E}(G',x)$ where y is conjugate to x in G^* (in particular $\mathrm{T}_{G'}^G Y_{M,(y)} = (0)$ in case that

G'^* contains no conjugate of y). Now we can refine Equation 4.8 to

$$\mathrm{T}^G_{G'} Y_{G,(y)} = \bigoplus_{x \in C_{G'^*,(y)}} Y_{G',(x)}, \tag{4.10}$$

where $C_{G'^*,y}$ is a set of representatives of the G'^*-conjugacy classes making up the intersection of G'^* with the G^*-conjugacy class of y. In particular, the indecomposable direct summands of $\mathrm{T}^G_{G'} Y_{G,(y)}$ are contained in pairwise different blocks.

Notation 4.11 In the following let G be as above. Let $L \in \mathcal{L}_G$. The underlying algebraic groups $\mathbf{G}$ and $\mathbf{L}$ have connected centers. We fix a semisimple element $z \in L^*$ and assume that we have a cuspidal element χ_L in $\mathcal{E}(L, z)$.

For $M \in \mathcal{L}_{G,L}$ such that M^* contains the semisimple Element $y \in G^*$ we set $X_{M,(y)} = \pi^M_{(L,\chi_L)}(Y_{M,(y)})$.

Recall the action of W on the set of pairs $(M^*, (y))$, where $M^* \in \mathcal{L}_G$, $y \in M^*$ is semi simple and (y) denotes its conjugacy class in M^*. Then $\mathrm{stab}_W(L^*)$ acts on the set of L^*-conjugacy classes of semi simple elements. In the following $\mathrm{stab}_W(z)$ denotes the stabilizer of (z) under this action. We denote the ℓ'-part of any semisimple element $y \in G^*$ by y'.

Theorem 4.12 *Suppose that χ_L is a regular character, and that $\mathrm{stab}_W(z) = \mathrm{stab}_W(z')$. Then $\mathcal{PR}(X_{G,(z)}, Y_{L,(z)})$ is a projective restriction system for (L, χ_L) in G consisting of the data*

$$\mathcal{PR}(X_{G,(z)}, Y_{L,(z)}) = \{X_M, Y_M, X'_M \mid M \in \mathcal{L}_{G,L}\},$$

where $X_M = X_{M,(z)}$, and Y_M and X'_M are defined as in 2.3.

Proof By 4.7 we only have to show part (ii) of Definiton 2.3.

As usual we denote the character of $X_G = X_{G,(z)}$ by χ_G. Let $M \in \mathcal{L}_{G,L}$ and $\chi_1 \neq \chi_2$ two summands of $\mathrm{T}^G_M \chi_G$. By 4.5 we are done if we can show that χ_1 and χ_2 are in two different unions of series defined in 4.6 and hence in different blocks. So assume that this is not the case. We may assume that both χ_1 and χ_2 are in the collection $\mathcal{E}_\ell(M, z')$ of blocks of M.

By Lemma 4.3 all summands of $\mathrm{T}^G_L \chi_G$ and hence of $\mathrm{T}^M_L \chi_i$ $(i = 1, 2)$ are in series $\mathcal{E}(L, y)$ for $y \in L^*$ conjugate to z in G^*. Moreover, by Equation 4.4 and general HC-theory, both $\mathrm{T}^M_L \chi_1$ and $\mathrm{T}^M_L \chi_2$ actually do have summands in in $\mathcal{E}_\ell(L, z')$ as the latter is the union of the series $\mathcal{E}(L, y)$ with $y' = z'$. Thus $\mathrm{T}^G_L \chi_G$ has at least two summands in $\mathcal{E}_\ell(L, z')$. However, by Mackey decomposition and cuspidality of χ_L,

$$\mathrm{T}^G_L \chi_G \leq \mathrm{R}^G_L \mathrm{T}^G_L \chi_L = \bigoplus_{x \in \mathrm{stab}_W(L)/(W \cap L)} \chi^x_L.$$

By Frobenius reciprocity, χ_L has multiplicity one in $\mathrm{T}^G_L \chi_G$, hence it follows that there exists $x \in \mathrm{stab}_W(L)$ such that

$$\chi_L \neq \chi^x_L \in \mathcal{E}_\ell(L, z'). \tag{4.13}$$

Now we extend Equation 4.4 to the union of series $\mathcal{E}_\ell(L, z')$. As the two unions of series $\mathcal{E}_\ell(L, z')$ and $\mathcal{E}_\ell(L, z'^x)$ are either equal or disjoint it follows

$$\mathcal{E}_\ell(L, z') = \mathcal{E}_\ell(L, z')^x = \mathcal{E}_\ell(L, z'^x),$$

hence $x \in \mathrm{stab}_W(z')$. By assumption we then also have $x \in \mathrm{stab}_W(z)$. Thus

$$\chi_L^x \in \mathcal{E}(L, z)^x = \mathcal{E}(L, z^x) = \mathcal{E}(L, z).$$

However, by construction (see [2, chapter 8]), Γ_L is invariant under the action of $\mathrm{stab}_W(L)$, hence χ_L^x is regular. As there is only one regular character in $\mathcal{E}(L, z)$, namely χ_L, we conclude $\chi_L^x = \chi_L$, a contradiction to 4.13. □

Note that the assumption of the previuous theorem fits precisely the situation considered in [10, section 5]. There the Hecke-algebra part for $\mathrm{R}_L^G X_L$ was done, and our theorem here provides its extension to q-Schur algebras. For the general linear groups $G = GL_n(q)$ this was done in [14]. In fact it was shown there, that we get a complete list of the irreducible ℓ-modular representations applying our main results 3.8,4.12 and Result 1.4 to the Lusztig series $\mathcal{E}(G, z)$, where z runs through a set of representatives of ℓ-regular conjugacy classes of $G = GL_n(q)$. Thus we obtain from Results 2.40 and 3.9 a new proof for the following results from [4],[5],[37] and [14] (compare [5, 4.15, 6.11]):

Theorem 4.14 *Let $G = GL_n(q)$. Then every cuspidal irreducible KG-module remains irreducible if reduced modulo ℓ. On the other hand, every irreducible cuspidal kG-module is liftable to an irreducible KG-module. If L is a Levi subgroup of G, and X_L is an irreducible cuspidal $\mathcal{O}L$-lattice such that $\mathrm{R}_L^G(X_L)$ is reduction stable then the combined image $\tau_{P_G}(\mathrm{R}_L^G X_L)$ of the Gelfand-Graev representation in $\mathrm{R}_L^G X_L$ is pure in $\mathrm{R}_L^G X_L$.*

These results can be generalized to classical groups under the additional assumption that the prime ℓ is linear. For details we refer to [28]. Here is another situation, where our results apply:

Corollary 4.15 *Suppose that χ_L is regular and $\bar{\chi}_L$ is irreducible. Moreover, suppose that the stabilizers $\mathrm{stab}_W(\chi_L)$ and $\mathrm{stab}_W(\bar{\chi}_L)$ of χ_L and $\bar{\chi}_L$ under the action of $\mathrm{stab}_W(L)$ are equal. Then $\mathcal{PR}(X_{G,(z)}, Y_{L,(z)})$ is a projective restriction system for (L, χ_L).*

Proof χ_L^x for $x \in \mathrm{stab}_W(L)$ is regular (see the last proof), hence it is the unique regular character in $\mathcal{E}(L, z^x)$ and $\mathrm{stab}_W(z) \leq \mathrm{stab}_W(\chi_L)$. As the converse follows from Equation 4.4, we have equality. Since there is a bijection between the set of irreducible Brauer characters of kL and representatives of isomorphism classes of indecomposable projective $\mathcal{O}L$-lattices, we see that $\mathrm{stab}_W(\bar{\chi}_L)$ is equal to the stabilizer of $Y_{L,(z')}$ under the action of $\mathrm{stab}_W(L)$ (analogeously denoted by $\mathrm{stab}_W(Y_{L,(z')})$). As P_L is invariant under this action and has exactly one indecomposable direct summand in every union of series $\mathcal{E}_\ell(L, z')$, we get $\mathrm{stab}_W(Y_{L,(z')}) = \mathrm{stab}_W(z')$ and the assertion follows from Theorem 4.12. □

In the situation of Theorem 4.12 as well as of Corollary 4.15 we have exhibited projective restriction systems $\mathcal{PR}(X_G, Y_L)$, which allow us to apply our main results of the previous section. In both cases the Hecke algebra $H = \mathrm{End}_{\mathcal{O}G}(\mathrm{R}_L X_L)$ is a Hecke algebra associated with an extension of a reflection group and it is known that it admits an associative (in fact symmetric) bilinear form, whose determinant is a unit, hence is integrally Frobenius, ([10] in the first and [24] in the second case). Thus Theorem 3.13 implies:

Corollary 4.16 *The decomposition matrix of the q-Schur algebra $\mathcal{S}_{\mathcal{O}}^G = \mathcal{S}_{\mathcal{O}}^G(\mathcal{PR})$ is part of the ℓ-modular decomposition matrix of G.*

We assume now that G is a Levi subgroup of some larger group of Lie type $\tilde{G}$. Again we assume that the center of $\tilde{\mathbf{G}}$ is connected.

Theorem 4.17 *Suppose that $\mathcal{E}(L,z)$ has only one cuspidal element χ_L and that $z = z'$. Let χ_G be a summand of $\mathrm{R}_L^G\chi_L$ of multiplicity one. Suppose further that there exists some indecomposable projective $\mathcal{O}G$-lattice Y_G such that the character of $X_G = \pi^G_{(L,\chi_L)}(Y_G)$ is equal to χ_G. Then there exists an $\mathcal{O}L$-lattice X_L with character χ_L and projective cover Y_L such that $\mathcal{PR}(X_G, Y_L)$ is a projective restriction system for (L, χ_L) in $\tilde{G}$.*

Proof First we show part (i) of Definiton 2.3. To do so we show that if $L^g = L$ with $g \in \tilde{G}$ but $L^g \not\sim_G L$ then the character of Y_G has no summand in common with $\mathrm{R}^G_{L^g}\chi_L^g$. Recall that we can restrict attention to standard Levi subgroups, hence we can assume $g \in \tilde{N}$, respectively we can take $g = w$ in the Weyl group W of $\tilde{G}$. Then by 4.4

$$\chi_L^w \in \mathcal{E}(L,z)^w = \mathcal{E}(L^w, z^w)$$

(using the action of W on the conjugacy classes of $\tilde{G}^*$ via $W \cong W^*$) and hence

$$\mathrm{R}^G_{L^g}\chi_L^g \in \mathcal{E}(G, z^w) \subset \mathcal{E}_\ell(G, z^w).$$

However, $z = z'$, hence if z^w is not conjugate to z in G then we have for the set of blocks $\mathcal{E}_\ell(G,z^w) \cap \mathcal{E}_\ell(G,z) = \emptyset$. As Y_G is indecomposable, we are done.

For part (ii) of Definiton 2.3 we show that the summands of $\mathrm{T}^G_M\chi_G$ for $M \in \mathcal{L}_{G,L}$ are in pairwise different blocks. So assume this is not the case, that is let χ_1, χ_2 be summands of $\mathrm{T}^G_M\chi_G$ in the same block. Again we may assume that one of them is $\chi_M \in \mathcal{E}(M,z)$. By 4.2, every summand of $\mathrm{T}^G_M\chi_G$ lies in some series $\mathcal{E}(M,y) \subset \mathcal{E}_\ell(M,y)$ for $y \in M^*$ conjugate to z in G^*. As $z = z'$, it follows then by [1, 2.2] that both χ_1 and χ_2 are in $\mathcal{E}(M,z)$. Like in the proof of Theorem 4.12 we conclude that both of $\mathrm{T}^M_L\chi_i$ has a summand in $\mathcal{E}(L,z)$. However, by general HC-theory, these summands must be cuspidal and thus by our assumption they must be equal to χ_L. Thus χ_L has multiplicity of at least two in $\mathrm{T}^G_L\chi_G$. By Frobenius reciprocity this gives a contradiction to our assumption. □

Remark 4.18 If L is a group of classical type, every Lusztig series has at most one cuspidal character (see [2, 13]).

The following corollary now follows immediately from the preceeding sections

Corollary 4.19 *Let the assumptions be as in Theorem* 4.17. *Then the q-Schur algebra* $\mathcal{S}_{\mathcal{O}}^{\tilde{G}}(\mathcal{PR}(X_G, Y_L))$ *is defined and its decomposition matrix is a submatrix of the decomposition matrix of* $\mathcal{E}(\tilde{G}, z)$.

An example of particular interest for the previous two Theorems are the regular characters in series $\mathcal{E}(L, y)$ for $y = y'$. We call such series ℓ-regular Lusztig series. Note that if some Lusztig series has just one element then this element is a regular character.

There are two main examples of the above, which have been investigated in previous papers by various authors. We keep the outline of the following examples sketchy, for further detailes the reader may refer to the cited references.

4.20 The first one is the well known theory for finite general linear groups. It was developed in several articles by G. James and the first named author (see [6], [4], [5], [14], [37] and [38]). First one observes that all the Hecke algebras appearing as endomorphism rings of induced irreducible cuspidal KL-modules for $L \in \mathcal{L}_G$, are defined over groups H isomorphic to direct products of symmetric groups. Secondly, any irreducible cuspidal KL-module for $L \in \mathcal{L}_G$ is regular. Therefore one can apply Theorem 4.12 to cuspidal characters in ℓ-regular Lusztig series. It follows that the decomposition matrix of the q-Schur algebra $\mathcal{S}_{\mathcal{O}}(\mathcal{PR}(X_{G,(z)}, Y_{L,(z)}))$ gives a part of the decomposition matrix of $\mathcal{E}(G, z)$, where z is ℓ-regular and $L \in \mathcal{L}_G$ is a minimal Levi subgroup containing z (for G a general linear group, we have $G \cong G^*$ canonically). Moreover, it turns out that $\mathcal{S}_{\mathcal{O}}(\mathcal{PR}(X_{G,(z)}, Y_{L,(z)}))$ has the same number of isomorphism classes of projective indecomposable modules as $\mathcal{S}_K(\mathcal{PR}(X_{G,(z)}, X_{L,(z)}))$ (see [14]). However, the latter are by definition in bijection to the elements $S(G/L, \chi_L)$, where χ_L is the unique (cuspidal) character in $\mathcal{E}(L, z)$. As HC-series and Lusztig series for finite general linear groups coincide, it follows that the decomposition numbers of the various q-Schur algebras defined over cuspidal irreducible regular characters in ℓ-regular Lusztig series gives us the complete decomposition matrix of all series $\mathcal{E}(G, z)$ for ℓ-regular elements $z \in G^* = G$. The decomposition matrix of the other characters can now easily deduced by decomposition of Deligne Lusztig induced characters. However, the results in this article can be applied to an arbitrary group G such that $\mathbf{G}$ is of type A with connected center. Thus it follows that the decomposition numbers of the ℓ-regular Lusztig series of G are the same as the decomposition numbers of ℓ-regular Lusztig series of suitable general linear groups.

4.21 The second example concernes other finite classical groups $\tilde{G}$ such as symplectic, special orthogonal and general unitary groups. They were investigated in [28] by G. Hiss and the second named author. If one restricts attention to so called linear primes for these groups (see [28], Section 1 for definition, and note that linear primes are roughly half of all primes dividing the order of $\tilde{G}$), one can develop a theory similar to the one of finite general linear groups.

First one observes that the reduction modulo ℓ of every cuspidal character χ_L of a Levi subgroup L of $\tilde{G}$ is irreducible.

Secondly, let $\phi \in \mathcal{E}(L, z), \psi \in \mathcal{E}(M, s)$ for $L, M \in \mathcal{L}_{\tilde{G}}$ and $z, s \in \tilde{G}^*$ ℓ-regular, be two cuspidal irreducible characters not conjugate in N. Then two characters $\alpha \in S(\tilde{G}/L, \phi)$ and $\beta \in S(\tilde{G}/M, \psi)$ are lying in different blocks.

Thirdly, for every cuspidal irreducible character in an ℓ-regular Lusztig series $\mathcal{E}(L, z)$ for some $L \in \mathcal{L}_{\tilde{G}}$ one can define a projective restriction system and an associated q-Schur algebra of type B or D. Using the representation theory of Hecke algebras over classical extended Weyl groups for linear primes (see [13] and [28], Section 7) one again finds that the q-Schur algebra $\mathcal{S}_{\mathcal{O}}(\mathcal{PR}(X_G, Y_L))$ has the same number of isomorphism classes of projective indecomposable modules as the q-Schur algebra over K (see [28] Corollary 8.2). Thus we again have the complete decomposition matrix of all series $\mathcal{E}(\tilde{G}, z)$ for ℓ-regular elements $z \in \tilde{G}^*$.

In these examples, Y_L is the unique indecomposable projective $\mathcal{O}L$-lattice affording a character with some irreducible cuspidal constituent $\chi_L \in \mathcal{E}(L, z)$, while X_G is a suitably chosen $\mathcal{O}G$-lattice affording a character χ_G with multiplicity one in $\mathrm{R}_L^G \chi_L$. $G \in \mathcal{L}_{\tilde{G}}$ is chosen as follows: The Dynkin diagram of the Weyl group of L has at most one connected component I of type B or D (if it has no such component we set $I = \emptyset$). Then G is a maximal Levi subgroup of $\tilde{G}$ containing L such that its Weyl group has one connected component of type A and one equal to I.

We note that in both examples one can apply Corollary 2.40 to the projective restriction systems associated to irreducible cuspidal characters in ℓ-regular Lusztig series. We get a complete description of the module structure of the so called Steinberg lattice, that is the unique quotient lattice of the Gelfand-Graev lattice affording the Steinberg character. In case of general linear groups, the corresponding quotient of the Gelfand-Graev lattice affording other irrducible characters (even for those characters not lying in ℓ-regular Lusztig series) are described in [26]. There, also for the special linear groups corresponding results have been obtained.

4.22 Let $\tilde{\mathbf{G}}$ be an arbitrary reductive algebraic group with connected center. For simplicity of notation we assume that the Dynkin diagram of $\tilde{\mathbf{G}}$ is connected. However, the following can be formulated more generally. Let $L \in \mathcal{L}_{\tilde{G}}$ have a unique cuspidal character χ_L in the series $\mathcal{E}(L, z)$ for some $z \in L^*$. Let $\hat{X}$ be a KL-module with character χ_L. The Dynkin diagram of L has at most one connected component $\mathcal{I}$ which is not of type A. Let $\mathcal{J}$ be the union of the connected components of the Dynkin diagram of L of type A. Then there exists a maximal Levi subgroup $G \in \mathcal{L}_{\tilde{G}}$ whose Dynkin diagram has at most two connected components, one being of type A and having $\mathcal{J}$ as subset and one equal to $\mathcal{I}$. Then $\mathrm{stab}_W(\chi_L)$ is a direct product of symmetric groups. We take χ_G to be the character in $S(G/L, \chi_L)$ corresponding to the alternating character of $\mathrm{stab}_W(\chi_L)$. In particular, $\pi_{(L,\chi_L)}(\mathrm{T}_M^G \chi_G) = \chi_M$ is irreducible for $M \in \mathcal{L}_{G,L}$ and $\pi_{(L,\chi_L)}(\mathrm{T}_L^G \chi_G) = \chi_L$ by [31], Theorem 5.9. Finally suppose that there exists an indecomposable projective $\mathcal{O}G$-lattice Y_G such that $\pi_{(L,\chi_L)}(Y_G) = X_G$ has character χ_G. Setting $X_M = \pi_{(L,\chi_L)}(\mathrm{T}_M^G X_G)$ for $M \in \mathcal{L}_{G,L}$ and denoting the projctive cover of X_M by Y_M, we can apply Theorem 4.17 to get a projective restriction system $\mathcal{PR}(X_G, Y_L)$. Moreover, because of our assumptions, we can apply Corollary 4.19. Obviously, if $L \cong G' \times L_1$ and $G \cong G' \times L_2$, where G' is a finite group of Lie type with Dynkin diagram $\mathcal{I}$ (as it holds in the case of finite classical groups), it follows from the theory of linear groups that Y_G exists. Moreover, if L is of classical type, there always exists at most one cuspidal character in a Lusztig series of L. We can formulate the following

Corollary 4.23 *Let the notation and assumptions be as in 4.22. Then the q-Schur algebra* $\mathcal{S}_{\mathcal{O}}(\mathcal{PR}(X_G, Y_L))$ *is defined and its decomposition matrix embeds into the decomposition matrix of* $\mathcal{E}(\tilde{G}, z)$.

We want to discuss an application of our theory that has not been investigated in other articles, yet, the decomposition matrix of the unipotent characters of the general unitary groups for arbitrary odd primes. To do so we need some preliminary considerations about the general linear groups.

4.24 For the following Lemma the reader might recall the labeling of irreducible unipotent characters of the general linear group $G = GL_n(q^2)$ by partitions $\alpha \vdash n$ of n (see [2, 13.8]). Here, a unipotent Brauer character is a Brauer character appearing as summand of the reduction modulo ℓ of some unipotent character. Using the lexicographic order for partitions (see [37]), the Steinberg character has the lowest label. For any prime $\ell \neq p$, the decomposition matrix of the unipotent characters, arranged with respect to the lexicographic order along a vertical edge of the decomposition matrix, can be made unitriangular if the unipotent Brauer characters are arranged suitably on a horizontal edge of the decomposition matrix (see again [37]). Moreover, the decomposition matrix of the unipotent characters is a square matrix. Giving the i^{th} unipotent Brauer character the label of the i^{th} unipotent character, we get a labeling of the unipotent Brauer characters by the partitions of n. We remark that square unitriangularity of the decomposition matrix determines the labeling of the Brauer characters uniquely. This follows easily from the fact that a different labeling of the Brauer characters has the effect of multiplying the decomposition matrix with a permutation matrix and if the resulting matrix is again square unitriangular then the permutation matrix is necessarily the unit matrix.

Lemma 4.25 *Let $\mu \neq (1^n)$ be a partition of n. Then there exists an odd prime ℓ and some power q of $p \neq \ell$ such that the unipotent Brauer character ϕ_μ of $GL_n(q^2)$ with label μ appears in the reduction modulo ℓ of at least two different irreducible unipotent characters of G. Moreover, ℓ can be chosen to be linear for $GU_n(q)$.*

Proof The proof proceeds in two steps. First we prove the assertion for $\mu = (j, 1^{n-j})$ for $j \geq 1$. If n is odd, take ℓ to be a Zsigmondy prime of $q^n - 1$ (see [36, 8.3]) for some q an arbitrary power of some prime p. The order of q^2 modulo ℓ is the same as the order of q modulo ℓ and ℓ is linear for $GU_n(q)$. Now the assertion follows in this case immediately from [37, 6.5] by taking $e = n$ in that theorem. For $n = 2$ the the assertion follows from [37, 6.5] using $\ell = 3$ and $q = 4$. Now let $n > 2$ be even. Let q be an arbitrary power of some prime p (in case $n = 6$ take $q \neq 2$). Let ℓ be a Zsigmondy prime of $q^{2n} - 1$. Then ℓ is again linear for $GU_n(q)$ and the assertion follows in this case from [37, 6.5] by taking $e = 2n$. The second step is to proceed by induction. Let $\mu = (\mu_1, \dots, \mu_r)$. We set $\mu' = (\mu_2, \dots, \mu_r) \vdash n - \mu_1$. We can assume that $(0) \neq \mu' \neq (1^{n-\mu_1})$. By induction hypothesis there exists $\ell \neq p$, a power q of p and a partition $\lambda' \neq \mu'$ of $n - \mu_1$ such that the ℓ-modular unipotent Brauer character of $GL_{n-\mu_1}(q^2)$ with label μ' is a constituent of the reduction modulo ℓ of the unipotent $KGL_{n-\mu_1}(q^2)$-character with label $\lambda' = (\lambda_2, \dots, \lambda_s)$ and ℓ is linear for $GU_{n-\mu_1}(q)$. Moreover, by unitriangularity of the decomposition matrix it follows that $\lambda' \leq \mu'$ in the lexicographical order. Thus $\mu_1 \geq \lambda_i$ for $2 \leq i \leq s$. By [37, 6.18] ϕ_μ is a constituent of the reduction modulo ℓ of the unipotent $KGL_n(q)$-character with label $(\mu_1, \lambda_2, \dots, \lambda_s)$. □

4.26 For the remainder of this section let G be the finite general unitary group over the field with q^2 elements and let $\ell \neq 2$. We will investigate unipotent characters, thus we assume that χ_L is a cuspidal unipotent irreducible character for some $L \in \mathcal{L}_G$. We remark however that the following can also be applied to some other ℓ-regular Lusztig series $\mathcal{E}(G,z)$ by 4.22, as the Levi subgroups of G are direct products of at most one unitary factor and general linear groups over $\mathbb{F}_{q^2}$. The characters in $\mathcal{E}(G,z)$ are canonically labeled by the unipotent characters of $C_G(z)$ and their decomposition matrices are the same (see [28]). By [2, 13.8] the irreducible unipotent KG-characters can be labeled by partitions of n. Similarly to the linear groups, it was shown in [23, 6.6] that the decomposition matrix of G is square lower unitriangular (for appropriate arrangements of the characters). We again get a labeling of the unipotent Brauer characters by partitions of n like in the case of $GL_n(q)$. Now the unipotent characters of G (and thus the Brauer characters) can also be labeled by the bipartitions of $\{(n-r)/2\}_r$, where r runs through all positive integers less or equal to n such that $n-r$ is even and the unitary group of degree r has a unipotent cuspidal character (see [28], for the relevant r see [2, 13.7]). Here a bipartition of n is an ordered pair of partitions $\alpha \vdash a$ and $\beta \vdash b$ with $a+b=n$. A unipotent character has as label some bipartition of $(n-r)/2$ if and only if it lies in the HC-series of the cuspidal irreducible unipotent character χ_L of the Levi subgroup $L \cong GU_r(q) \times GL_1(q^2)^{(n-2)/2}$ of G (see [2, 13.8]).

Now consider $S(G/L, \chi_L)$. Then we have

Theorem 4.27 *Let ψ be the character in $S(G/L,\chi_L)$ with least label in the lexicographical order. Then ψ appears with multiplicity one in $\mathrm{R}_L^G\chi$.*

Proof We turn to the labeling of the characters in $S(G/L,\chi)$ by the bipartitions of $s=(n-r)/2$. Suppose that ψ has label (α,β) for partitions $\alpha \vdash a$ and $\beta \vdash b = s-a$. By the assumption on ψ, it follows from the lower unitriangularity of the decomposition matrices that there exists no odd prime ℓ' and no power q' of some prime p' different to ℓ' such that the ℓ'-modular irreducible unipotent Brauer character with the same label as ψ appears as constituent of the reduction modulo ℓ of some element in $S(G'/L',\chi')$ different to ψ. Here G' denotes $GU_n(q')$, L' the a standard Levi subgroup of G' isomorphic to $GU_r(q') \times GL_1({q'}^2)^{(n-2)/2}$ and χ' the cuspidal irreducible unipotent character of G'. Recall that by general theory the multiplicities of summands ρ' in $\mathrm{R}_{L'}^{G'}\chi'$ and ρ in $\mathrm{R}_L^G\chi_L$ are the same, if ρ and ρ' have the same label. Now assume that $\alpha \neq (1^a)$. Then choose ℓ' and q' with Lemma 4.25 such that the ℓ'-modular irreducible unipotent Brauer character of $GL_a({q'}^2)$ with label α appears in the reduction modulo ℓ' of some irreducible unipotent character with label $\gamma \neq \alpha$ and ℓ' is linear for $GU_a(q')$. Linearity of ℓ' only depends on q', thus ℓ' is linear for $GU_n(q')$. Now it follows from [25, 3.36] that the ℓ-modular unipotent Brauer character of $GU_n(q')$ is a constituent of the reduction modulo ℓ of the character in $S(G'/L',\chi'_L)$ with label (γ,δ), a contradiction to our statement at the beginning of the proof. Thus $\alpha=(1^a)$. Similarly one shows that $\beta=(1^b)$.

Now recall the way how one passes from labeling of the characters in $\mathcal{E}(G,1)$ by partitions of n to their labeling by bipartitions. This way is described for example in [18]. If $r=0$, the partition (1^n) is the label of the Steinberg character and the least partition in the lexicographical order. Thus we may assume that $r>0$. Assume that $a>0$ and $b>0$.

Then one can see immediately, that if λ is the partition labeling ψ, then λ must be of one of the following forms.

$$a)\quad (m+3+b, m+3+(b-1), \ldots, m+3, m, m-1, \ldots, 2, 1^{2a+1}),$$

$$b)\quad (2+b, 2+(b-1), \ldots, 3, 1^{2a}),$$

$$c)\quad (3+b, 3+(b-1), \ldots, 4, 1^{2a+1}),$$

for $m \geq 2$. However, consider the partition $\delta = (m+b+1, \ldots, 2, 1^{2a+2b+1})$ in case $a)$, respectively $\delta = (b, \ldots, 2, 1^{2a+2b+1})$ in cases $b)$ and $c)$. Then δ is a partition of n and corresponds to a bipartition of the form $((0), (1^s))$ or $((1^s), (0))$. Moreover, δ is lower in the lexicographical order than λ, a contradiction to our assumption on ψ. Thus it follows that $a = 0$ or $b = 0$. Now the assertion follows from the representation theory of the Weyl group of type B, see [2]. □

As a consequence of Proposition 4.27 we can apply Theorem 4.17 to get our last theorem. We use the following notation for it. Let χ_L be an irreducible cupidal unipotent character of $L \in \mathcal{L}_G$. Let χ_G be the character of lowest label in $S(G/L, \chi_L)$. Let ϕ be the unipotent Brauer character of G with the same label as χ_G and let Y_G be an indecomposable projective $\mathcal{O}G$-lattice such that $Y_G/\mathrm{Jac}Y_G$ has Brauer character ϕ. Let $X_M = \pi_{(L,\chi_L)}(\mathrm{T}_M^G Y_G)$ for $M \in \mathcal{L}_{G,L}$, let χ_M be its character and let Y_M be the projective cover of X_M.

Theorem 4.28 *The data $\{(\chi_M, X_M, Y_M) \mid M \in \mathcal{L}_{G,L}\}$ define a projective restriction system $\mathcal{PR}(X_G, Y_L)$. In particular, the decomposition matrix of $\mathcal{S}_{\mathcal{O}}(\mathcal{PR}(X_G, Y_L))$ is a submatrix of the decomposition matrix of $\mathcal{E}(G, 1)$.*

Proof By square unitriangularity of the decomposition matrix of $\mathcal{E}(G,1)$ it follows that ϕ is not a constituent of the reduction modulo ℓ of any character in $S(G/L, \chi_L)$ different to χ_L and its multiplicity in $\bar{\chi}_L$ is one. Thus $\pi_{(L,\chi_L)}(Y_G) = X_G$ has character χ_G. Now the first statement follows from Theorem 4.17. The second statement follows from Corollary 4.19. □

4.29 We have so far only considered finite Lie groups whose underlying algebraic group has connected center. In case of classical groups it was shown in [28] that one can extend the results for linear primes to groups whose underlying algebraic group has non-connected center. In general this is done by embedding the group in a finite Lie groups whose underlying algebraic group has conncted center. In case of classical groups the center of the underlying algebraic group has two connected components. We get more problems in case the number of connected components increase. As a standard example for this to happen we can consider the finite special linear groups. These groups were treated in [26]. There it was shown how one can describe the decomposition matrix of a set of irreducible characters whose reduction modulo ℓ generate the group of generalized Brauer characters in terms of decomposition matrices of q-Schur algebras defined over extended Weyl groups of type A.

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