Morita equivalences of Ariki–Koike algebras

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Was lange währt, wird endlich gut.

Abstract

We prove that every Ariki–Koike algebra is Morita equivalent to a direct sum of tensor products of smaller Ariki–Koike algebras which have q–connected parameter sets. A similar result is proved for the cyclotomic q–Schur algebras. Combining our results with work of Ariki and Uglov, the decomposition numbers for the Ariki–Koike algebras defined over fields of characteristic zero are now known in principle.

1 Introduction

Let R be a commutative ring with 1 and let $q, Q_1, \dots, Q_r$ be elements of R with q invertible. Let $\mathbf{Q} = (Q_1, Q_2, \dots, Q_r)$. The Ariki–Koike algebra $\mathscr{H}_{q,\mathbf{Q}}(n)$ is the associative unital R–algebra with generators $T_0, T_1, \dots, T_{n-1}$ subject to the following relations

$$\begin{array}{rcll}(T_0 - Q_1)\dots(T_0 - Q_r) & = & 0 & \\ T_0T_1T_0T_1 & = & T_1T_0T_1T_0 & \\ (T_i + 1)(T_i - q) & = & 0 & \text{for } 1 \le i \le n-1 \\ T_{i+1}T_iT_{i+1} & = & T_iT_{i+1}T_i & \text{for } 1 \le i \le n-2 \\ T_iT_j & = & T_jT_i & \text{for } 0 \le i < j-1 \le n-2.\end{array}$$

The main result of this paper states that up to Morita equivalence the Ariki–Koike algebras depend only on the q–orbits of the parameters in $\mathbf{Q}$. More precisely we have the following.

1.1 Theorem *Suppose that* $\mathbf{Q} = \mathbf{Q}_1 \coprod \mathbf{Q}_2 \coprod \dots \coprod \mathbf{Q}_\kappa$ *(disjoint union) is a partitioning* Π *of the parameter set* $\mathbf{Q}$ *such that*

$$f_\Pi(q, \mathbf{Q}) = \prod_{1 \le \alpha < \beta \le \kappa} \prod_{\substack{Q_i \in \mathbf{Q}_\alpha \\ Q_j \in \mathbf{Q}_\beta}} \prod_{-n < a < n} (q^a Q_i - Q_j)$$

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is an invertible element of R. Then $\mathscr{H}_{q,\mathbf{Q}}(n)$ is Morita equivalent to the algebra

$$\mathscr{H}_{q,\Pi}(n) = \bigoplus_{\substack{n_1,\dots,n_\kappa\geq 0\\ n_1+\dots+n_\kappa=n}} \mathscr{H}_{q,\mathbf{Q}_1}(n_1)\otimes\mathscr{H}_{q,\mathbf{Q}_2}(n_2)\otimes\dots\otimes\mathscr{H}_{q,Q_\kappa}(n_\kappa).$$

Notice that the polynomial $f_\Pi(q,\mathbf{Q})$ is invertible only if whenever there exist i and j with $q^aQ_i = Q_j$ for some a with $-n<a<n$ then $Q_i, Q_j\in\mathbf{Q}_\alpha$ for some α. The appearance of these polynomials is not surprising because it was shown by Ariki [**1**] that over a field $\mathscr{H}$ is semisimple if and only if

$$P_{\mathscr{H}}(q,\mathbf{Q}) = \prod_{1\leq i<j\leq r}\prod_{-n<a<n}(q^aQ_i-Q_j)\cdot\prod_{k=1}^n(1+q+\dots+q^{k-1})\neq 0.$$

The polynomial $P_{\mathscr{H}}(q,\mathbf{Q})$ is the analogue of the Poincaré polynomial for $\mathscr{H}$. The factors of $P_{\mathscr{H}}(q,\mathbf{Q})$ in the right hand product determine whether or not the subalgebra $\mathscr{H}_q(\mathfrak{S}_n)$ is semisimple.

Permuting the parameters $Q_1,\dots,Q_r$ does not affect $\mathscr{H}$ up to isomorphism. In addition, if $\mathbf{Q}' = c\mathbf{Q} = (cQ_1,\dots,cQ_r)$, where c is any invertible element of R, then $\mathscr{H}_{q,\mathbf{Q}}(n)$ and $\mathscr{H}_{q,\mathbf{Q}'}(n)$ are isomorphic (replace T_0 with $c^{-1}T_0$). We now rephrase Theorem 1.1 so that it says that up to Morita equivalence $\mathscr{H}$ depends only on the q–orbits of $Q_1,\dots,Q_r$. To state this precisely, say that $\mathbf{Q}$ is q–connected if $Q_i = q^{a_i}$ for some integer a_i for each i (for the purposes of Corollary 1.2 we could require that $|a_i|<n$, for each i, but this is not so important). Normally, $\mathbf{Q}$ will not be q–connected; however, up to a permutation of the $\mathbf{Q}_\alpha$'s, there is a unique partitioning $\mathbf{Q} = \mathbf{Q}_1\coprod\dots\coprod\mathbf{Q}_\kappa$ such that for $\alpha = 1,\dots,\kappa$ there exists an element $c_\alpha\in R$ such that $Q_i\in\mathbf{Q}_\alpha$ if and only if $Q_i = c_\alpha q^{a_i}$ for some integer a_i; so, $\mathbf{Q}_\alpha = c_\alpha\mathbf{Q}'_\alpha$ for each α. By the above remarks, if c_α is invertible then $\mathbf{Q}'_\alpha$ is q–connected and $\mathscr{H}_{q,\mathbf{Q}_\alpha}(\mathfrak{S}_n)$ is isomorphic to $\mathscr{H}_{q,\mathbf{Q}'_\alpha}(\mathfrak{S}_n)$; so Theorem 1.1 implies the following.

1.2 Corollary *Suppose that each Q_i is invertible for $1\leq i\leq r$. Then the Ariki–Koike algebra $\mathscr{H}_{q,\mathbf{Q}}(n)$ is Morita equivalent to a direct sum of tensor products of Ariki–Koike algebras which have q–connected parameter sets.*

The importance of this result stems from the work of Ariki [**2**] which showed that if R is a field of characteristic zero, $q\neq 1$ and the parameter set $\mathbf{Q}$ is q–connected then the decomposition numbers of Ariki–Koike algebras can be computed in terms of the canonical basis of an associated integral highest weight module for an affine quantum group. In addition, Ariki [**4**] and Ariki and the second author [**6**] used the results of [**2**] to classify the irreducible representations of the Ariki–Koike algebras without any restrictions on the field, q or $\mathbf{Q}$; this was done by first reducing to the case of q–connected parameter sets. Corollary 1.2 explains the reduction of [**6**] by showing that it comes from a Morita equivalence.

In fact, we can do more than this because Uglov [**26**] (extending the ideas of [**23**]), gave an algorithm for computing the decomposition matrices of the

Ariki–Koike algebras which satisfy the restrictions of Ariki's paper [**2**]. In the spirit of the Kazhdan–Lusztig conjectures, Uglov's algorithm involves computing certain affine parabolic Kazhdan–Lusztig polynomials and evaluating them at 1. Combining [**26**] with Proposition 4.11(iii) below we obtain the following.

1.3 Corollary *Suppose that R is a field of characteristic zero, $q \neq 1$ and $Q_i \neq 0$ for $1 \leq i \leq r$. Then the decomposition matrix of $\mathscr{H}_{q,\mathbf{Q}}(n)$ is known.*

Next consider the case where there exists an integer a with $Q_i \neq q^a Q_j$ if and only if $i = j$; then Theorem 1.1 says that $\mathscr{H}_{q,\mathbf{Q}}(n)$ is Morita equivalent to a direct sum of tensor products of Hecke algebras of type A. This special case is a result of Du and Rui [**17**, Theorem 4.14].

1.4 Corollary (Du–Rui) *Suppose that $\prod_{1\leq i<j\leq r}\prod_{-n<a<n}(q^aQ_i - Q_j)$ is an invertible element of R. Then $\mathscr{H}_{q,\mathbf{Q}}(n)$ is Morita equivalent to*

$$\bigoplus_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\dots+n_r=n}} \mathscr{H}_q(\mathfrak{S}_{n_1}) \otimes \mathscr{H}_q(\mathfrak{S}_{n_2}) \otimes \cdots \otimes \mathscr{H}_q(\mathfrak{S}_{n_r}).$$

In fact, when $r = 2$ this is a theorem of James and the first named author [**15**, Theorem 4.17]. Although this paper is largely motivated by [**15**], the techniques we use are very different. As in [**15**] we explicitly construct the projective generator of $\mathscr{H}$ which induces the Morita equivalence of the main theorem; however, unlike [**15**, **17**] we do this by adapting the standard basis of $\mathscr{H}$ (from [**16**]) to give bases for the family of projective modules which describe the projective generator. This yields precise information, such as Specht series for these modules, which is new even in the special cases considered previously [**15**, **17**]. Another consequence is that we are able to extend our results to the cyclotomic q–Schur algebras $\mathscr{S}_{q,\mathbf{Q}}(n)$ of [**16**].

1.5 Theorem *Suppose that $\mathbf{Q} = \mathbf{Q}_1 \coprod \mathbf{Q}_2 \coprod \cdots \coprod \mathbf{Q}_\kappa$ (disjoint union) is a partitioning Π of the parameter set $\mathbf{Q}$ such that $f_\Pi(q,\mathbf{Q})$ is an invertible element of R. Then $\mathscr{S}_{q,\mathbf{Q}}(n)$ is Morita equivalent to the algebra*

$$\mathscr{S}_{q,\Pi}(n) = \bigoplus_{\substack{n_1,\dots,n_\kappa\geq 0\\ n_1+\dots+n_\kappa=n}} \mathscr{S}_{q,\mathbf{Q}_1}(n_1) \otimes \mathscr{S}_{q,\mathbf{Q}_2}(n_2) \otimes \cdots \otimes \mathscr{S}_{q,\mathbf{Q}_\kappa}(n_\kappa).$$

Actually, a slightly more general statement is possible; see Theorem 5.2.

Again some special cases of Theorem 1.5 were known previously. First, if $r = 2$ then $\mathscr{H}$ is an Iwahori–Hecke algebra of type B and in this case Theorem 1.5 can be deduced for special parameter sets from the results of [**10**, **20**]. The case $r = 2$ is important because it has implications for representation theory of symplectic and unitary groups; see [**8**, **9**]. Secondly, Ariki [**3**] (see also Du and Rui [**17**]), have proved Theorem 1.5 under the assumptions of Corollary 1.4; here the cyclotomic q–Schur algebra arises as a quotient of a quantum group of type A acting on "q–tensor space" as the centralizing algebra of the Ariki–Koike algebra.

In order to prove Theorem 1.1 we first observe that it is enough to prove the following much simpler, but equivalent, result.

1.6 Theorem *Fix an integer s with $1 \le s \le r$ and suppose that*

$$f_s(q,\mathbf{Q}) = \prod_{1\le i\le s<j\le r} \prod_{-n<a<n} (q^a Q_i - Q_j)$$

is an invertible element of R. Then $\mathscr{H}_{q,\mathbf{Q}}$ is Morita equivalent to

$$\mathscr{H}_{q,s,\mathbf{Q}}(n) = \bigoplus_{b=0}^{n} \mathscr{H}_{q,(Q_{s+1},\dots,Q_r)}(\mathfrak{S}_b) \otimes \mathscr{H}_{q,(Q_1,\dots,Q_s)}(\mathfrak{S}_{n-b}).$$

The general case follows by iterating Theorem 1.6, using the remarks before Corollary 1.2.

The proof of Theorem 1.6 is based on an explicit decomposition of $\mathscr{H} = \mathscr{H}_{q,\mathbf{Q}}(n)$ into a direct sum of projective right ideals V^b using the standard basis of $\mathscr{H}$. The different projective modules V^b have no direct summand in common and $V = \bigoplus_{b=0}^n V^b$ is a progenerator for $\mathscr{H}$. In Theorem 3.20 we give an explicit formula for the multiplicity of each V^b in the regular representation of $\mathscr{H}$ and in Theorem 4.7 we show that the endomorphism ring of V is $\mathscr{H}_{q,s,\mathbf{Q}}(n)$; this proves Theorem 1.6 and that the Morita equivalences of Theorem 1.6 are given by the functors $-\otimes_{\mathscr{H}_{q,s,\mathbf{Q}}} V$ and $\operatorname{Hom}_{\mathscr{H}}(V,-)$. The explicit description of these functors enables us to extend our results to the cyclotomic q-Schur algebras in section 5.

2 The standard basis theorem

Let $\mathfrak{S}_n$ be the symmetric group on $\{1,2,\dots,n\}$, acting from the right, and let $s_1,\dots,s_{n-1}$ be the standard Coxeter generators of $\mathfrak{S}_n$; that is, $s_i = (i,i+1)$ for all i. If $w \in \mathfrak{S}_n$ write $w = s_{i_1}s_{i_2}\dots s_{i_k}$ and say this expression is reduced if k is minimal; in this case, k is the length of w and we write $\ell(w) = k$ and define $T_w = T_{i_1}T_{i_2}\dots T_{i_k}$. Let $\mathscr{H}(\mathfrak{S}_n)$ be the R–span of $\{\, T_w \mid w \in W\,\}$; then $\mathscr{H}(\mathfrak{S}_n)$ is a free subalgebra of $\mathscr{H}$ of rank $n!$ which is isomorphic to the Iwahori–Hecke algebra of $\mathfrak{S}_n$. The Iwahori–Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ is described in detail in [**24**].

Let $L_1 = T_0$ and for $1 \le i < n$ set $L_{i+1} = q^{-1}T_iL_iT_i$. These elements satisfy the following fundamental relations (see [**5**, (3.3)] and [**16**, (2.1)]).

2.1 *Suppose that $1 \le i \le n-1$ and $1 \le j \le n$. Then*
 (i) *L_i and L_j commute.*
 (ii) *T_i and L_j commute if $i \ne j-1, j$.*
 (iii) *T_i commutes with L_iL_{i+1} and with $L_i + L_{i+1}$.*
 (iv) *If $a \in R$ and $i \ne j$ then T_i commutes with $(L_1-a)(L_2-a)\dots(L_j-a)$.*

The importance of these elements derives from the following result.

2.2 (Ariki–Koike [5, Theorem 3.10]) *The algebra $\mathscr{H}$ is free as an R–module with basis*

$$\{\, L_1^{d_1} L_2^{d_2} \dots L_n^{d_n} T_w \mid w \in \mathfrak{S}_n \text{ and } 0 \le d_m \le r-1 \text{ for } m = 1, 2, \dots, n \,\}.$$

In particular, $\mathscr{H}$ is free of rank $r^n n!$

Note that because q is invertible so are the elements T_i, for $i = 1, \dots, n-1$; explicitly, $T_i^{-1} = q^{-1}(T_i - q + 1)$. Consequently, T_w is invertible for all $w \in \mathfrak{S}_n$.

Let $* : \mathscr{H} \longrightarrow \mathscr{H}$ be the anti–automorphism of $\mathscr{H}$ determined by $T_i^* = T_i$ for $i = 0, 1, \dots, n-1$. Then $T_w^* = T_{w^{-1}}$ for all $w \in \mathfrak{S}_n$ and $L_i^* = L_i$ for $i = 1, 2, \dots, n$.

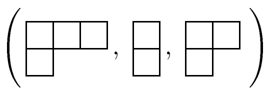
As we next recall, the Ariki–Koike algebra has another basis which is better adapted to the study of its representation theory; this basis is cellular, in the sense of Graham and Lehrer [18]. (Graham and Lehrer were the first to construct a cellular basis of $\mathscr{H}$; the basis we use is due to Gordon James and the authors [16].)

A composition of an integer $m \ge 0$ is an ordered sequence of non–negative integers $\tau = (\tau_1, \tau_2 \dots)$ such that $|\tau| = \sum_{i \ge 1} \tau_i = m$; if the sequence is non–increasing then τ is a partition of m. A multicomposition of n (with r–components) is an ordered r–tuple $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ of compositions such that $|\mu^{(1)}| + \dots + |\mu^{(r)}| = n$. If each $\mu^{(i)}$ is a partition then μ a multipartition. Let $\Lambda = \Lambda(n\,{:}\,r)$ be the set of multicompositions of n with r–components and let $\Lambda^+ = \Lambda^+(n\,{:}\,r) \subset \Lambda$ be the set of multipartitions of n with r–components.

The diagram of a multicomposition μ is the set

$$[\mu] = \{\, (i, j, k) \mid 1 \le k \le r \text{ and } i \ge 1 \text{ and } 1 \le j \le \mu_i^{(k)} \,\},$$

which we think of as an ordered r–tuple of boxes in the plane. For example, if $\mu = ((3,1), (1^2), (2,1))$ then

$[\mu] =$  .

In this way, we talk of the rows and columns (of the components) of μ.

Given two multicompositions λ and μ say that λ dominates μ, and write $\lambda \trianglerighteq \mu$, if for $1 \le c \le r$ and for all $i \ge 1$

$$\sum_{b=1}^{c-1} |\lambda^{(b)}| + \sum_{j=1}^{i} \lambda_j^{(c)} \ge \sum_{b=1}^{c-1} |\mu^{(b)}| + \sum_{j=1}^{i} \mu_j^{(c)}.$$

If $\lambda \trianglerighteq \mu$ and $\lambda \ne \mu$ we write $\lambda \triangleright \mu$. This defines a partial order on the sets of multicompositions and multipartitions of n.

If μ is a multicomposition of n then a μ–tableau is a map $\mathfrak{t}:[\mu]\longrightarrow\{1,2\dots,n\}$; we write $\operatorname{Shape}(\mathfrak{t})=\mu$. Generally, we shall think of tableaux as labelled diagrams; for example, when $\mu=((3,1),(1^2),(2,1))$ three μ–tableaux are

$$\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array},\ \begin{array}{|c|c|}\hline 7&8\\\hline \multicolumn{1}{|c|}{9}\\\cline{1-1}\end{array}\right),\quad \left(\begin{array}{|c|c|c|}\hline 3&5&8\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 2\\\hline 9\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&6\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}\right)\quad\text{and}\quad \left(\begin{array}{|c|c|c|}\hline 3&5&8\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 7\\\hline 6\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{9}\\\cline{1-1}\end{array}\right).$$

Given a tableau $\mathfrak{t}$ we will also write $\mathfrak{t}=(\mathfrak{t}^{(1)},\dots,\mathfrak{t}^{(r)})$ and call $\mathfrak{t}^{(i)}$ the ith component of $\mathfrak{t}$. Similarly, given an integer m with $1\le m\le n$ we write $\operatorname{comp}_{\mathfrak{t}}(m)=i$ if m appears in the ith component $\mathfrak{t}^{(i)}$ of $\mathfrak{t}$. If $\mathfrak{s}$ is another tableau we write $\operatorname{comp}_{\mathfrak{t}}=\operatorname{comp}_{\mathfrak{s}}$ if $\operatorname{comp}_{\mathfrak{t}}(m)=\operatorname{comp}_{\mathfrak{s}}(m)$ for all m with $1\le m\le n$.

A μ–tableau $\mathfrak{t}$ is standard if the entries in each row of each component of $\mathfrak{t}$ increase from left to right and the entries in each column of each component of $\mathfrak{t}$ increase from top to bottom (of the tableaux above only the first two are standard). For each multipartition λ let $\operatorname{Std}(\lambda)$ be the set of standard λ–tableaux.

Suppose $\mathfrak{t}$ is a standard λ–tableaux and $\mathfrak{s}$ a standard μ–tableau for $\lambda,\mu\in\Lambda$. Given an integer m with $1\le m\le n$ let $\mathfrak{t}{\downarrow}m$ be the subtableau of $\mathfrak{t}$ which contains the entries $1,2,\dots,m$ and similarly for $\mathfrak{s}{\downarrow}m$. Then $\mathfrak{s}\trianglerighteq\mathfrak{t}$ if $\operatorname{Shape}(\mathfrak{s}{\downarrow}m)\trianglerighteq\operatorname{Shape}(\mathfrak{t}{\downarrow}m)$, for $m=1,2,\dots,n$, and we say that $\mathfrak{s}$ dominates $\mathfrak{t}$. Again we write $\mathfrak{s}\triangleright\mathfrak{t}$ if $\mathfrak{s}\trianglerighteq\mathfrak{t}$ and $\mathfrak{s}\ne\mathfrak{t}$.

Let $\mathfrak{t}^\mu$ be the unique μ–tableau such that $\mathfrak{t}^\mu\trianglerighteq\mathfrak{t}$ for all μ–tableau $\mathfrak{t}$. Then $\mathfrak{t}^\mu$ is the tableau which has the numbers $1,2,\dots,n$ entered in order along the rows of $[\mu]$. Note that the symmetric group $\mathfrak{S}_n$ acts from the right on the set of μ–tableaux. Let $\mathfrak{S}_\mu$ be the row stabilizer of the tableau $\mathfrak{t}^\mu$; then $\mathfrak{S}_\mu$ is a parabolic subgroup of $\mathfrak{S}_n$. In the example above, where $\mu=((3,1),(1^2),(2,1))$, the first of the tableaux listed is $\mathfrak{t}^\mu$ and $\mathfrak{S}_\mu=\mathfrak{S}_3\times\mathfrak{S}_1\times\mathfrak{S}_1\times\mathfrak{S}_1\times\mathfrak{S}_2\times\mathfrak{S}_1\hookrightarrow\mathfrak{S}_9$ (obvious embedding); we will always identify $\mathfrak{S}_\mu$ with a subgroup of $\mathfrak{S}_n$ in this way.

For each μ–tableau $\mathfrak{t}$ let $d(\mathfrak{t})$ be the unique element of $\mathfrak{S}_n$ such that $\mathfrak{t}=\mathfrak{t}^\mu d(\mathfrak{t})$. Then, by [**11**, Lemma 1.4], $d(\mathfrak{t})$ is a distinguished right coset representative of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$; that is, $\ell(wd(\mathfrak{t}))=\ell(w)+\ell(d(\mathfrak{t}))$ for all $w\in\mathfrak{S}_\mu$.

Given an r–tuple $\mathbf{a}=(a_1,a_2,\dots,a_r)$ of integers, with $0\le a_i\le n$ for all i, let $u_{\mathbf{a}}=u_{a_1,1}u_{a_2,2}\dots u_{a_r,r}$ where $u_{a,t}=\prod_{k=1}^{a}(L_k-Q_t)$ for any a and t. If μ is a multicomposition of n let $u_\mu^+=u_{\mathbf{a}}$ where $\mathbf{a}=(a_1,a_2,\dots,a_r)$ is the sequence with $a_t=|\mu^{(1)}|+\dots+|\mu^{(t-1)}|$ for $t=1,2,\dots,r$.

Suppose that λ is a multipartition of n and let $x_\lambda=\sum_{w\in\mathfrak{S}_\lambda}T_w$ and set $m_\lambda=u_\lambda^+x_\lambda$. If $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ–tableaux define $m_{\mathfrak{st}}=T^*_{d(\mathfrak{s})}m_\lambda T_{d(\mathfrak{t})}$. Then we have the following.

2.3 (Dipper–James–Mathas [**16**, Theorem 3.26]**)** *Let*

$$\mathcal{M}=\{\,m_{\mathfrak{st}}\mid\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\lambda)\ \textit{for some}\ \lambda\in\Lambda^+\,\}.$$

Then $\mathcal{M}$ is a cellular basis of $\mathscr{H}$.

The basis $\mathcal{M}$ is called the standard basis of $\mathscr{H}$.

As in [**16**], let N^λ be the R–module with basis the set of all $m_{\mathfrak{st}} \in \mathcal{M}$ where $(\mathfrak{s},\mathfrak{t})$ runs over all pairs of standard μ–tableaux with $\mu \trianglerighteq \lambda$; similarly, let $\overline{N}^\lambda$ be the R–module with basis the set of $m_{\mathfrak{uv}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableau and $\mu \triangleright \lambda$. From the theory of cellular algebras we obtain the following corollary of (2.3).

2.4 ([**16**, Corollary 3.22]) *The R–modules N^λ and $\overline{N}^\lambda$ are two–sided ideals of $\mathscr{H}$.*

Given a multipartition λ the Specht module S^λ is the submodule of $\mathscr{H}/\overline{N}^\lambda$ defined by $S^\lambda = z_\lambda\mathscr{H}$ where $z_\lambda = m_\lambda + \overline{N}^\lambda$. The theory of cellular algebras [**18**, **24**] shows that S^λ is free of rank $|\operatorname{Std}(\lambda)|$ (with basis $\{\, m_{\mathfrak{t}^\lambda\mathfrak{t}} + \overline{N}^\lambda \mid \mathfrak{t} \in \operatorname{Std}(\lambda)\,\}$), and that there is an intrinsically defined symmetric $\mathscr{H}$–invariant bilinear form $(\ ,\ )$ on S^λ; that is $(uh, v) = (u, vh^*)$ for $u, v \in S^\lambda$ and $h \in \mathscr{H}$. Let $\operatorname{rad} S^\lambda$ be the radical of this form and set $D^\lambda = S^\lambda/\operatorname{rad} S^\lambda$. Then D^λ is an $\mathscr{H}$–module; moreover, the following is true.

2.5 [**16**,**18**] *Suppose that R is a field.*
(i) *If λ is a multipartition of n then D^λ is either (0) or absolutely irreducible.*
(ii) *$\{\, D^\lambda \mid \lambda \in \Lambda^+$ and $D^\lambda \neq (0)\,\}$ is a complete set of pairwise non–isomorphic irreducible $\mathscr{H}$–modules.*

Given multipartitions λ and μ with $D^\mu \neq (0)$ let $d_{\lambda\mu} = [S^\lambda : D^\mu]$ be the decomposition multiplicity of the simple module D^μ in the Specht module S^λ. The matrix $(d_{\lambda\mu})$ is the decomposition matrix of $\mathscr{H}$. Importantly, the decomposition matrix of $\mathscr{H}$ is unitriangular; more precisely, we have the following.

2.6 [**16**,**18**] *Suppose that R is a field and that λ and μ are multipartitions such that $D^\mu \neq (0)$. Then $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \trianglerighteq \mu$.*

Let $\mathfrak{s}$ be a tableau and suppose that $1 \le k \le n$ appears in row i and column j of the cth component $\mathfrak{s}^{(c)}$ of $\mathfrak{s}$. Then the residue of k in $\mathfrak{s}$ is $\operatorname{res}_\mathfrak{s}(k) = q^{j-i}Q_c$. The following result underpins much of what follows.

2.7 [**22**, Prop. 3.7] *Let $\mathfrak{s}$ and $\mathfrak{t}$ be standard λ–tableaux, where $\lambda \in \Lambda^+$, and suppose that $1 \le k \le n$. Then there exist $a_\mathfrak{u} \in R$ such that*

$$L_k m_{\mathfrak{st}} \equiv \operatorname{res}_\mathfrak{s}(k) m_{\mathfrak{st}} + \sum_{\mathfrak{u} \triangleright \mathfrak{s}} a_\mathfrak{u} m_{\mathfrak{ut}} \mod \overline{N}^\lambda.$$

For each multipartition λ define its content to be the sequence $\operatorname{cont}(\lambda) = (c_r)_{r\in R}$ where c_r is the number of nodes x of the diagram $[\lambda]$ with $\operatorname{res}(x) = r$. By [**18**, Theorem 3.7(ii)] all of the irreducible constituents of S^λ belong to the same block. Furthermore, by (2.1) every symmetric polynomial $f(L)$, in $L_1, L_2, \dots, L_n$ belongs to the centre of $\mathscr{H}$; therefore, by Schur's Lemma, $f(L)$ acts on S^λ as multiplication by a scalar, say $\alpha_f \in R$. By (2.7), $f(L)m_\lambda \equiv \alpha_f m_\lambda \mod \overline{N}^\lambda$; so it follows that α_f depends only upon the content of λ. Hence, we have the following result.

2.8 Corollary (Graham–Lehrer [18, Prop. 5.9(ii)]) *Suppose that λ and μ are multipartitions of n. Then S^λ and S^μ belong to the same block only if* $\operatorname{cont}(\lambda) = \operatorname{cont}(\mu)$.

Grojnowski [19] has recently shown that S^λ and S^μ are in the same block if and only if $\operatorname{cont}(\lambda) = \operatorname{cont}(\mu)$. (The definition of residue must be modified slightly in the case $q = 1$.) When $r = 1$ this result was already known by [12,21].

For each multicomposition μ of n let $M^\mu = m_\mu\mathscr{H}$. To describe how the standard basis of (2.3) can be modified to give a basis of M^μ we need to generalize the notion of tableau. Suppose that λ is a multipartition and μ is a multicomposition of n. A λ–tableau of type μ is a map $\mathtt{S}\colon[\lambda] \longrightarrow \{1,\dots,n\}\times\{1,\dots,r\}$ such that, for all (i,k), $\mu_i^{(k)} = \#\{\, x\in[\lambda] \mid \mathtt{S}(x) = (i,k)\,\}$; as before, we will think of $\mathtt{S} = (\mathtt{S}^{(1)},\dots,\mathtt{S}^{(r)})$ as a labelling of $[\lambda]$ with ordered pairs of integers (i,k). A λ–tableau $\mathtt{S}$ of type μ is semistandard if for $c = 1\dots,r$ the entries in the cth component $\mathtt{S}^{(c)}$ of $\mathtt{S}$ are (i) non–decreasing along the rows; (ii) strictly increasing down the columns and, (iii) no entry in $\mathtt{S}^{(c)}$ has the form (i,k) with $k<c$. Let $\mathcal{T}_0(\lambda,\mu)$ be the set of semistandard λ–tableaux of type μ.

Given a standard λ–tableau $\mathfrak{s}\colon[\lambda]\longrightarrow\{1,\dots,n\}$ define $\mu(\mathfrak{s})$ to be the λ–tableau of type μ obtained from $\mathfrak{s}$ by replacing each entry m in $\mathfrak{s}$ by (i,k), if m appears in row i of the kth component of $\mathfrak{t}^\mu$. Observe that in general the entries in each column of $\mu(\mathfrak{s})$ will only be non–decreasing so that $\mu(\mathfrak{s})$ need not be semistandard.

2.9 For example, if we let $\omega = \big((0),\dots,(0),(1^n)\big)$ then the tableau $\omega(\mathfrak{s})$ has entries of the form (i,r) where $1\le i\le n$. Consequently, $\mathfrak{s}\longmapsto\omega(\mathfrak{s})$ gives a bijection from the set of standard λ–tableaux to the set of semistandard λ–tableaux of type ω. Henceforth, we identify λ–tableau of type ω with the λ–tableaux that are maps from $[\lambda]$ to $\{1,\dots,n\}$; we also use lower case letters $\mathfrak{s},\mathfrak{t},\dots$ to denote λ–tableaux (of type ω) and upper case letters $\mathtt{S},\mathtt{T},\dots$ for tableaux of arbitrary type.

Before we can state the basis theorem for M^μ we need one more definition. Let $\mathtt{S}$ be a semistandard tableau of type μ and a λ–tableau $\mathfrak{t}$; set

$$m_{\mathtt{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathtt{S}}} m_{\mathfrak{s}\mathfrak{t}}.$$

2.10 [16, Theorem 4.14] *Suppose that μ is a multicomposition of n. Then M^μ is free as an R–module with basis*

$$\{\, m_{\mathtt{S}\mathfrak{t}} \mid \mathtt{S}\in\mathcal{T}_0(\lambda,\mu), \mathfrak{t}\in\operatorname{Std}(\lambda) \textit{ for some } \lambda\in\Lambda^+\,\}.$$

The last result that we shall need gives a basis for $\operatorname{Hom}_{\mathscr{H}}(M^\nu,M^\mu)$. Given a semistandard λ–tableau $\mathtt{S}$ of type μ and a semistandard tableau $\mathtt{T}$ of type ν let

$$m_{\mathtt{S}\mathtt{T}} = \sum_{\substack{\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathtt{S},\nu(\mathfrak{t})=\mathtt{T}}} m_{\mathfrak{s}\mathfrak{t}}$$

and define $\varphi_{\mathtt{ST}} : M^\nu \longrightarrow M^\mu$ to be the $\mathscr{H}$–module homomorphism given by $\varphi_{\mathtt{ST}}(m_\nu h) = m_{\mathtt{ST}} h$ for all $h \in \mathscr{H}$. It is not completely obvious that $\varphi_{\mathtt{ST}}$ even belongs to $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\nu)$; nonetheless, the following is true.

2.11 [**16**, Theorem 6.6(i)] *Suppose that μ and ν are multicompositions of n. Then $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\nu)$ is free as an R–module with basis*

$$\{\, \varphi_{\mathtt{ST}} \mid \mathtt{S} \in \mathcal{T}_0(\lambda,\mu) \text{ and } \mathtt{T} \in \mathcal{T}_0(\lambda,\nu) \text{ for some } \lambda \in \Lambda^+ \,\}.$$

3 A projective generator for $\mathscr{H}$

For the remainder of the paper we fix an integer s with $1 \le s \le r$.

Following [**15**], given integers i and j with $1 \le i < j < n$ define

$$s_{i,j} = s_i s_{i+1} \dots s_{j-1} \quad \text{and} \quad s_{j,i} = s_{i,j}^{-1} = s_{j-1} s_{j-2} \dots s_i.$$

Note that $s_{j,i}$ is the cycle $(i, i+1, \dots, j)$. We abbreviate the corresponding elements of $\mathscr{H}$ by $T_{i,j} = T_{s_{i,j}}$ and $T_{j,i} = T_{s_{j,i}}$; so, $T_{j,i} = T^*_{i,j}$. In passing, we remark that $L_i = q^{1-i} T_{i,1} T_0 T_{1,i}$ for $i = 1, \dots, n$.

If a and b are non–negative integers we define $w_{a,b} = (s_{a+b,1})^b$; in particular, $w_{a,0} = w_{0,b} = 1$. Written as a permutation,

$$w_{a,b} = \begin{pmatrix} 1 & 2 & \dots & a & a+1 & a+2 & \dots & a+b \\ b+1 & b+2 & \dots & a+b & 1 & 2 & \dots & b \end{pmatrix}.$$

For later use we note that $w_{a,b}^{-1} = w_{b,a}$; consequently, $T^*_{w_{a,b}} = T_{w_{b,a}}$.

3.1 Lemma *Let a and b be non–negative integers with $0 \le a+b \le n$ and suppose that i is an integer such that $i \ne a$ and $1 \le i < a+b$. Then*

$$T_i T_{w_{a,b}} = T_{w_{a,b}} T_{(i)w_{a,b}} = \begin{cases} T_{w_{a,b}} T_{i+b}, & \text{if } 1 \le i < a, \\ T_{w_{a,b}} T_{i-a}, & \text{if } a < i < a+b. \end{cases}$$

Proof For any $w \in \mathfrak{S}_n$ we have $s_i w = w w^{-1} s_i w = w(iw, (i+1)w)$; setting $w = w_{a,b}$ we find

$$s_i w_{a,b} = \begin{cases} w_{a,b} s_{b+i}, & \text{if } 1 \le i < a, \\ w_{a,b} s_{i-a}, & \text{if } a < i < a+b. \end{cases}$$

The result follows by observing that $\ell(s_i w_{a,b}) = \ell(w_{a,b}) + 1$ since $w_{a,b}$ is a distinguished right coset representative of $\mathfrak{S}_{(a,b)}$ in $\mathfrak{S}_n$; see [**15**, 2.7]. □

Equation [**15**, 2.9] provides a recursive way to find a reduced expression for the permutation $w_{a,b}$. As a consequence we obtain the following Lemma.

3.2 Lemma *Suppose that* $1 \le b \le n-1$. *Then there exists* $\tilde{w} \in \mathfrak{S}_{(1,n-1)}$ *such that* $w_{n-b,b} = \tilde{w}s_{1,b+1}$ *and* $\ell(w_{n-b,b}) = \ell(\tilde{w}) + \ell(s_{1,b+1})$. *In particular,* $T_{w_{n-b,b}} = T_{\tilde{w}}T_{1,b+1}$.

Proof By [**15**, 2.9] if $a \ge 0$ and $0 \le a+b < n$ then $w_{a,b} = s_{a,a+b}w_{a-1,b}$ and $\ell(w_{a,b}) = \ell(s_{a,a+b}) + \ell(w_{a-1,b})$. Therefore,

$$\begin{aligned} w_{n-b,b} &= s_{n-b,n}w_{n-b-1,b} = s_{n-b,n}s_{n-b-1,n-1}w_{n-b-2,b} = \dots \\ &= s_{n-b,n}s_{n-b-1,n-1}\dots s_{2,b+2}w_{1,b} = (s_{n-b,n}\dots s_{2,b+2})s_{1,b+1}, \end{aligned}$$

with the lengths adding throughout. The Lemma follows. □

For the next definition recall that we have fixed an integer s with $1 \le s \le r$. For each integer b with $0 \le b \le n$ define

$$u_{n-b}^- = \prod_{t=1}^{s}(L_1 - Q_t)(L_2 - Q_t)\dots(L_{n-b} - Q_t)$$

and

$$u_b^+ = \prod_{t=s+1}^{r}(L_1 - Q_t)(L_2 - Q_t)\dots(L_b - Q_t).$$

These elements are special instances of the elements $u_{\mathbf{a}}$ introduced in the previous section.

We now define the modules which are the cornerstone upon which this paper is built.

3.3 Definition *Suppose that* $0 \le b \le n$ *and define* $v_b = u_{n-b}^- T_{w_{n-b,b}} u_b^+$ *and let* $V^b = v_b\mathscr{H}$.

Ultimately we shall show that V^b is a projective $\mathscr{H}$–module and that its endomorphism ring is isomorphic to a tensor product of smaller Ariki–Koike algebras; this will imply our main result. At this point it is not even clear that v_b is non–zero; we will deduce this important fact latter. We begin by establishing some key properties of v_b.

3.4 Proposition *Suppose that* $0 \le b \le n$.

(i) *If* $1 \le i < n-b$ *then* $T_iv_b = v_bT_{i+b}$.
(ii) *If* $n-b < i \le n$ *then* $T_iv_b = v_bT_{i-n+b}$.
(iii) *If* $1 \le k \le n-b$ *then* $L_kv_b = v_bL_{k+b}$.
(iv) *If* $n-b+1 \le k \le n$ *then* $L_kv_b = v_bL_{k-n+b}$.

Proof First, observe that parts (i) and (ii) follow from parts (i) and (ii) of Lemma 3.1, respectively, together with (2.1)(iv).

Next, consider part (iii). If $b=0$ or $b=n$ then v_b is central in $\mathscr{H}$ by parts (ii) and (iii) of (2.1); so our claims follow in these two cases and we may assume that $1\le b\le n-1$.

We first consider the case $k=1$; that is, L_1v_b. We write $w_{n-b,b}=\tilde{w}s_{1,b+1}$, as in Lemma 3.2; then $\tilde{w}\in S_{(1,n-1)}$ so that L_1 and $T_{\tilde{w}}$ commute by (2.1). Now each L_i commutes with u^-_{n-b} since the L_i generate an abelian subalgebra of $\mathscr{H}$; therefore,

$$\begin{aligned}
L_1v_b &= L_1u^-_{n-b}T_{w_{n-b,b}}u^+_b\\
&= u^-_{n-b}L_1T_{\tilde{w}}T_{1,b+1}u^+_b\\
&= u^-_{n-b}T_{\tilde{w}}L_1T_{1,b+1}u^+_b\\
&= u^-_{n-b}T_{\tilde{w}}T^{-1}_{b+1,1}T_{b+1,1}L_1T_{1,b+1}u^+_b\\
&= q^bu^-_{n-b}T_{\tilde{w}}T^{-1}_{b+1,1}L_{b+1}u^+_b\\
&= q^bu^-_{n-b}T_{\tilde{w}}(T^{-1}_1\dots T^{-1}_b)u^+_bL_{b+1}.
\end{aligned}$$

Now $q^b(T^{-1}_1\dots T^{-1}_b)=(T_1-q+1)\dots(T_b-q+1)=T_1\dots T_b+h$, where $h\in\mathscr{H}$ is an R–linear combination of terms of the form T_xT_y such that (x,y) is an element of $\mathfrak{S}_i\times\mathfrak{S}_{b-i}=\mathfrak{S}_{(i,b-i)}$ for some i with $0<i<b$. Write $u^-_{n-b}=u_1\tilde{u}$, where $u_1=\prod_{t=1}^s(L_1-Q_t)$, and suppose $(x,y)\in\mathfrak{S}_{(i,b-i)}$ for some $i>0$. Then $T_xT_y=T_{xy}=T_{yx}=T_yT_x$ and, by (2.1), T_x commutes with u^+_b, T_y commutes with $\tilde{u}$ and $T_{\tilde{w}}$ commutes with u_1. Therefore,

$$u^-_{n-b}T_{\tilde{w}}T_{xy}u^+_b=u_1\tilde{u}T_{\tilde{w}}T_yT_xu^+_b=\tilde{u}T_{\tilde{w}}u_1T_yT_xu^+_b=\tilde{u}T_{\tilde{w}}T_yu_1u^+_bT_x=0;$$

the last equality following because $\prod_{t=1}^r(L_1-Q_t)=0$ is a factor of $u_1u^+_b$. Consequently, $q^bu^-_{n-b}T_{\tilde{w}}hu^+_bL_{b+1}=0$. Hence,

$$L_1v_b=u^-_{n-b}T_{\tilde{w}}(T_1\dots T_b)u^+_bL_{b+1}=u^-_{n-b}T_{w_{n-b,b}}u^+_bL_{b+1}=v_bL_{b+1}$$

as claimed.

Next consider L_kv_b for some k with $1<k\le n-b$. By induction and part (i),

$$L_kv_b=q^{-1}T_{k-1}L_{k-1}T_{k-1}v_b=q^{-1}T_{k-1}v_bL_{b+k-1}T_{b+k-1}=v_bL_{b+k},$$

proving (iii).

Part (iv) can be proved similarly; however, here is a better argument. As there is no essential difference between u^+_b and u^-_{n-b} — and hence between v_b and v^*_b — it follows from part (iii) that if $n-b<k\le n$ then $L_kv_b=(v^*_bL_k)^*=(L_{k-n+b}v^*_b)^*=v_bL_{k-n+b}$, giving (iv). □

Define $\mathsf{L}_A(m)=\{\,x\in A\mid xm=0\,\}$ to be the left annihilator of $m\in M$ in A; $\mathsf{L}_A(h)$ is a left ideal of A. We also let $\operatorname{Ann}_A(M)=\bigcap_{m\in M}\mathsf{L}_A(m)$ be the annihilator of M; this is an ideal of A. We will apply these definitions in the case where $M=V^b$ and $A=H_b$ is the subalgebra of $\mathscr{H}$ generated by $\{\,T_i,T_j,L_k\mid 1\le i<n-b<j<n \text{ and } 1\le k\le n\,\}$. Observe that by the Proposition V^b is invariant under left multiplication by H_b and hence a left H_b-module.

3.5 Corollary *Suppose that $0 \leq b \leq n$ and let the subalgebra H_b of $\mathscr{H}$ be defined as above. Then H_b acts on V^b by left multiplication and $\mathsf{L}_{\mathscr{H}}(v_b) \cap H_b = \operatorname{Ann}_{H_b}(V^b)$; consequently, the algebra $\hat{H}_b = H_b / \operatorname{Ann}_{H_b}(V^b)$ is a subalgebra of $\operatorname{End}_{\mathscr{H}}(V^b)$.*

In fact, in Theorem 4.7 below we will prove that

$$\hat{H}_b \cong \operatorname{End}_{\mathscr{H}}(V^b) \cong \mathscr{H}_{q,(Q_1,\dots,Q_s)}(b) \otimes \mathscr{H}_{q,(Q_{s+1},\dots,Q_r)}(n-b).$$

The next result shows that when L_{n-b+1} and L_1act on V^b by left multiplication they each satisfy one of the relations of the generators $1 \otimes T_0$ and $T_0 \otimes 1$, respectively, in the tensor product above.

3.6 Corollary *Suppose that $0 \leq b \leq n$. Then*
(i) $(L_1 - Q_{s+1}) \dots (L_1 - Q_r) v_b = 0$;
(ii) $(L_{n-b+1} - Q_1) \dots (L_{n-b+1} - Q_s) v_b = 0$;
(iii) $v_b (L_1 - Q_1) \dots (L_1 - Q_s) = 0$; *and,*
(iv) $v_b (L_{b+1} - Q_{s+1}) \dots (L_{b+1} - Q_r) = 0$.

Proof Parts (i) and (iii) follow from the relation $\prod_{t=1}^r (L_1 - Q_t) = 0$ and the definition of v_b; for parts (ii) and (iv) apply the Proposition 3.4 to parts (iii) and (i) respectively. □

3.7 Corollary *Let $0 \leq b < c \leq n$. Then $u_{n-b}^- T_{w_{n-b,b}} u_c^+ = 0$.*

Proof Let $h = \prod_{t=s+1}^r (L_{b+2} - Q_t) \cdots (L_c - Q_t)$. Then

$$\begin{aligned} u_{n-b}^- T_{w_{n-b,b}} u_c^+ &= u_{n-b}^- T_{w_{n-b,b}} \prod_{t=s+1}^r (L_1 - Q_t)(L_2 - Q_t) \cdots (L_c - Q_t) \\ &= u_{n-b}^- T_{w_{n-b,b}} u_b^+ (L_{b+1} - Q_{s+1}) \cdots (L_{b+1} - Q_r) h \\ &= v_b (L_{b+1} - Q_{s+1}) \cdots (L_{b+1} - Q_r) h = 0, \end{aligned}$$

where the last equality comes from Corollary 3.6(iv). □

We will study the ideals V^b by thinking of them as quotients of one of the modules M^μ. Let $\omega_b = (\omega_b^{(1)}, \dots, \omega_b^{(r)})$ be the multipartition of n with

$$\omega_b^{(t)} = \begin{cases} (1^b), & \text{if } t = s, \\ (1^{n-b}), & \text{if } t = r, \\ (0), & \text{otherwise.} \end{cases}$$

Then $u_b^+ = u_{\omega_b}^+ = m_{\omega_b}$; consequently, $v_b = u_{n-b}^- T_{w_{n-b,b}} m_{\omega_b}$ and V^b is a quotient of M^{ω_b}. This motivates the following definition.

3.8 Definition *Suppose that* $0 \le b \le n$. *Let* $\theta_b : M^{\omega_b} \longrightarrow V^b$ *be the map given by* $\theta_b(h) = u^-_{n-b} T_{w_{n-b,b}} h$ *for all* $h \in M^{\omega_b}$.

Thus, θ_b is a surjective $\mathscr{H}$–module homomorphism from M^{ω_b} onto V^b. The map θ_b is the main tool we need to understand the modules V^b; first we set up some notation.

Let λ be a multipartition of n and set

$$\mathrm{Std}_b(\lambda) = \{\, \mathfrak{t} \in \mathrm{Std}(\lambda) \mid \mathrm{comp}_{\mathfrak{t}}(k) \le s \text{ whenever } 1 \le k \le b \,\}\,;$$

that is, $\mathfrak{t} \in \mathrm{Std}_b(\lambda)$ if and only if the numbers $1, 2, \dots, b$ all appear in one of the first s components of $\mathfrak{t}$. Similarly, let

$$\mathrm{Std}_{b,n-b}(\lambda) = \{\, \mathfrak{t} \in \mathrm{Std}_b(\lambda) \mid \mathrm{comp}_{\mathfrak{t}}(k) > s \text{ whenever } b < k \le n \,\}\,.$$

Let $\Lambda^+_b = \{\, \lambda \mid \lambda \in \Lambda^+ \text{ and } |\lambda^{(1)}| + \cdots + |\lambda^{(s)}| = b \,\}$; then $\mathrm{Std}_{b,n-b}(\lambda)$ is non–empty if and only if $\lambda \in \Lambda^+_b$. On the other hand, $\mathrm{Std}_b(\lambda)$ is non–empty if and only if $|\lambda^{(1)}| + \cdots + |\lambda^{(s)}| \ge b$; we set

$$\overline{\Lambda}_b{}^+ = \{\, \lambda \mid \lambda \in \Lambda^+ \text{ and } |\lambda^{(1)}| + \cdots + |\lambda^{(s)}| > b \,\}\,.$$

Then $\overline{\Lambda}_b{}^+$ is a coideal in Λ^+; that is, if $\mu \in \Lambda^+$ and $\mu \trianglerighteq \lambda$ for some $\lambda \in \Lambda^+_b$ then $\mu \in \Lambda^+_b$. In contrast, Λ^+_b is not a coideal; however, $\Lambda^+_b \cup \overline{\Lambda}_b{}^+$ is a coideal and Λ^+_b and $(\Lambda^+_b \cup \overline{\Lambda}_b{}^+)/\overline{\Lambda}_b{}^+$ are isomorphic posets.

Observe that if $\lambda \in \Lambda^+_b$ then $\mathrm{Std}_{b,n-b}(\lambda) = \mathrm{Std}_b(\lambda)$; however, we will continue to write $\mathrm{Std}_{b,n-b}(\lambda)$ in order to emphasize the restrictions on the components of these tableaux.

3.9 Lemma *Suppose that* $0 \le b \le n$. *Then* M^{ω_b} *is free as an* R*–module with basis* $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s} \in \mathrm{Std}_b(\lambda)$ *and* $\mathfrak{t} \in \mathrm{Std}(\lambda)$ *for some* $\lambda \in \Lambda^+_b \cup \overline{\Lambda}_b{}^+ \,\}$.

Proof As noted in (2.10), M^{ω_b} is free as an R–module with basis $m_{\mathtt{S}\mathfrak{t}}$, where $\mathtt{S} \in \mathcal{T}_0(\lambda, \omega_b)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$ for some multipartition λ of n. It follows from the definitions that $\mathcal{T}_0(\lambda, \omega_b)$ is non–empty if and only if $\lambda \in \Lambda^+_b \cup \overline{\Lambda}_b{}^+$; further, if $\lambda \in \Lambda^+_b \cup \overline{\Lambda}_b{}^+$ then there is a bijection between $\mathrm{Std}_b(\lambda)$ and $\mathcal{T}_0(\lambda, \omega_b)$ given by $\mathfrak{s} \longmapsto \omega_b(\mathfrak{s})$ (*cf.* (2.9)). Consequently, if $\mathtt{S} \in \mathcal{T}_0(\lambda, \omega_b)$ then $\mathtt{S} = \omega_b(\mathfrak{s})$ for a uniquely determined $\mathfrak{s} \in \mathrm{Std}_b(\lambda)$ and $m_{\mathtt{S}\mathfrak{t}} = m_{\mathfrak{s}\mathfrak{t}}$. Combining these observations gives the result. □

Next we identify some elements in the kernel of θ_b; shortly we will see that these elements are actually a basis of $\ker \theta_b$.

3.10 Lemma *Suppose that* $0 \le b \le n$ *and* $\lambda \in \overline{\Lambda}_b{}^+$. *Let* $\mathfrak{s}$ *and* $\mathfrak{t}$ *be standard* λ*–tableaux with* $\mathfrak{s} \in \mathrm{Std}_b(\lambda)$. *Then* $\theta_b(m_{\mathfrak{s}\mathfrak{t}}) = 0$.

Proof Suppose $h = \theta_b(m_{\mathfrak{s}\mathfrak{t}}) \ne 0$. Let $c = |\lambda^{(1)}| + \cdots + |\lambda^{(s)}|$; then $\lambda \in \Lambda^+_c$ so that $b < c$. Now, because $\lambda \in \Lambda^+_c$ and $\mathfrak{s} \in \mathrm{Std}_b(\lambda)$, we can find a permutation $w \in \mathfrak{S}(\{b+1, \dots, n\})$ such that $\mathfrak{s}w \trianglerighteq \mathfrak{s}$ and $\mathfrak{s}w$ is a standard λ–tableau with the numbers $1, \ldots, c$ all appearing in the first s components of $\mathfrak{s}_w$. (If $a_1 < \cdots < a_k$ are the numbers between $b+1$ and c which appear in the last $r-s$ components of $\mathfrak{s}$ and $b_1 < \cdots < b_k$ are the k smallest numbers larger than b which appear in the first s components of $\mathfrak{s}$ then we can set $w = (a_1, b_1) \ldots (a_k, b_k)$.) Let $\mathfrak{s}_w = \mathfrak{s}w$; then $d(\mathfrak{s}) = d(\mathfrak{s}_w)w^{-1}$ and $m_{\mathfrak{s}\mathfrak{t}} = T_w m_{\mathfrak{s}_w\mathfrak{t}}$. Also let $\tilde{w} = w_{n-b,b}^{-1} w w_{n-b,b}$; then $\tilde{w} \in \mathfrak{S}_{n-b}$ and $T_{w_{n-b,b}} T_w = T_{\tilde{w}} T_{w_{n-b,b}}$ by Lemma 3.2. Therefore,

$$\theta_b(m_{\mathfrak{s}\mathfrak{t}}) = u_{n-b}^- T_{w_{n-b,b}} m_{\mathfrak{s}\mathfrak{t}} = u_{n-b}^- T_{w_{n-b,b}} T_w m_{\mathfrak{s}_w\mathfrak{t}} = u_{n-b}^- T_{\tilde{w}} T_{w_{n-b,b}} m_{\mathfrak{s}_w\mathfrak{t}}.$$

Now, $\tilde{w} \in \mathfrak{S}_{n-b}$ so $u_{n-b}^- T_{\tilde{w}} = T_{\tilde{w}} u_{n-b}^-$ by (2.1)(iv); consequently

$$\theta_b(m_{\mathfrak{s}\mathfrak{t}}) = T_{\tilde{w}} u_{n-b}^- T_{w_{n-b,b}} m_{\mathfrak{s}_w\mathfrak{t}} = T_{\tilde{w}} \theta_b(m_{\mathfrak{s}_w\mathfrak{t}}).$$

Since $T_{\tilde{w}}$ is invertible, $\theta_b(m_{\mathfrak{s}\mathfrak{t}}) \neq 0$ if and only if $T_{\tilde{w}}\theta_b(m_{\mathfrak{s}_w\mathfrak{t}}) \neq 0$; therefore, we may assume that $\mathfrak{s} = \mathfrak{s}_w$. Thus, it is enough to show that $\theta(m_{\mathfrak{s}\mathfrak{t}}) = 0$ whenever $\mathfrak{s} \in \mathrm{Std}_{c,n-c}(\lambda)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$. However, if $\mathfrak{s} \in \mathrm{Std}_{c,n-c}(\lambda)$ then $m_{\mathfrak{s}\mathfrak{t}} \in M^{\omega_c}$ by Lemma 3.9; so $m_{\mathfrak{s}\mathfrak{t}} = u_{\omega_c}^+ h$ for some $h \in \mathscr{H}$. Now $u_{\omega_c}^+ = u_c^+$; so $\theta_b(m_{\mathfrak{s}\mathfrak{t}}) = u_{n-b}^- T_{w_{n-b,b}} m_{\mathfrak{s}\mathfrak{t}} = u_{n-b}^- T_{w_{n-b,b}} u_c^+ h = 0$, by Corollary 3.7, as desired. □

Recall from Theorem 1.6 that given an integer s, with $1 \leq s \leq r$, we let

$$f_s(q, \mathbf{Q}) = \prod_{1 \leq i \leq s < j \leq r} \prod_{-n < a < n} (q^a Q_i - Q_j a).$$

For the remainder of this paper we assume that $f_s(q, \mathbf{Q})$ is an invertible element of R.

Suppose that $\mathfrak{s} \in \mathrm{Std}_{b,n-b}(\lambda)$ for some multipartition λ. Then $\mathfrak{s}' = \mathfrak{s}w_{b,n-b}$ is a standard λ–tableau which has the numbers $n-b+1, \ldots, n$ appearing in its first s components and the remaining numbers $1, \ldots, n-b$ appearing in its last $r-s$ components.

3.11 Lemma *Suppose that $\lambda \in \Lambda_b^+$ and let $\mathfrak{s}$ and $\mathfrak{t}$ be standard λ–tableaux with $\mathfrak{s} \in \mathrm{Std}_{b,n-b}(\lambda)$. Let $\mathfrak{s}' = \mathfrak{s}w_{b,n-b}$. Then*

(i) *$T_{w_{n-b,b}} m_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{s}'\mathfrak{t}}$; and,*
(ii) *there exists an invertible element $\alpha \in R$ such that*

$$\theta_b(m_{\mathfrak{s}\mathfrak{t}}) \equiv \alpha m_{\mathfrak{s}'\mathfrak{t}} + \sum_{\mathfrak{u}' \rhd \mathfrak{s}'} a_{\mathfrak{u}'} m_{\mathfrak{u}'\mathfrak{t}} \mod \overline{N}^\lambda,$$

for some $a_{\mathfrak{u}'} \in R$.

Proof (i) First note that $d(\mathfrak{s}) \in \mathfrak{S}_{(b,n-b)}$ and recall that $w_{b,n-b}$ is a distinguished right coset representative of $\mathfrak{S}_{(b,n-b)}$. Therefore, $\ell(d(\mathfrak{s}')) = \ell(d(\mathfrak{s})) + \ell(w_{b,n-b})$; consequently, $T_{w_{n-b,b}} m_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{s}'\mathfrak{t}}$ since $w_{n-b,b}^{-1} = w_{b,n-b}$.

(ii) Part (i) together with (2.7) implies that, modulo $\overline{N}^\lambda$,

$$\begin{aligned}\theta_b(m_{\mathfrak{st}}) &= u_{n-b}^- T_{w_{n-b,b}} m_{\mathfrak{st}} = \Big(\prod_{t=1}^{s}(L_1-Q_t)\dots(L_{n-b}-Q_t)\Big)m_{\mathfrak{s}'\mathfrak{t}}\\ &\equiv \Big(\prod_{t=1}^{s}(\operatorname{res}_{\mathfrak{s}'}(1)-Q_t)\dots(\operatorname{res}_{\mathfrak{s}'}(n-b)-Q_t)\Big)m_{\mathfrak{s}'\mathfrak{t}} + \sum_{\mathfrak{u}'\rhd\mathfrak{s}'} a_{\mathfrak{u}'}m_{\mathfrak{u}'\mathfrak{t}}\end{aligned}$$

for some $a_{\mathfrak{u}'}\in R$. Let $\alpha=\prod_{t=1}^{s}(\operatorname{res}_{\mathfrak{s}'}(1)-Q_t)\dots(\operatorname{res}_{\mathfrak{s}'}(n-b)-Q_t)$; then α is the coefficient of $m_{\mathfrak{s}'\mathfrak{t}}$ in $\theta_b(m_{\mathfrak{st}})$. Now, $1,2,\dots,n-b$ all belong to one of the last $r-s$ components of $\mathfrak{s}'$; so, for $1\le k\le n-b$, $\operatorname{res}_{\mathfrak{s}'}(k)=q^jQ_c$ for some c and j with $s<c\le r$ and $-n<j<n$. Therefore, α is a product of terms of the form $(q^jQ_c-Q_t)$, with $1\le t\le s$. As each of these factors divides $f_s(q,\mathbf{Q})$, it follows that α is invertible; so the Lemma is proved. □

Observe that the invertible element α in part (ii) of the Lemma depends only on $\lambda=\operatorname{Shape}(\mathfrak{s})$, rather than $\mathfrak{s}$ itself. Notice also that

$$v_b = u_{n-b}^- T_{w_{n-b,b}} m_{\omega_b} = u_{n-b}^- T_{w_{n-b,b}} m_{\mathfrak{t}^{\omega_b}\mathfrak{t}^{\omega_b}} \equiv \alpha m_{\mathfrak{t}\mathfrak{t}^{\omega_b}} \mod \overline{N}^\lambda,$$

where $\mathfrak{t}=\mathfrak{t}^{\omega_b}w_{b,n-b}$; in particular, we have finally proved that v_b is non–zero.

3.12 Definition *Suppose that* $0\le b\le n$ *and let* $\mathfrak{s}\in\operatorname{Std}_{b,n-b}(\lambda)$ *and* $\mathfrak{t}\in\operatorname{Std}(\lambda)$ *for some multipartition* $\lambda\in\Lambda_b^+$. *Let* $v_{\mathfrak{st}}=\theta_b(m_{\mathfrak{st}})=u_{n-b}^-T_{w_{n-b,b}}m_{\mathfrak{st}}$.

It follows from part (ii) of the Lemma that the $v_{\mathfrak{st}}$ are linearly independent elements in V^b. In fact, they are a basis of V^b.

3.13 Theorem *Suppose that* $f_s(q,\mathbf{Q})$ *is invertible in* R *and let* b *be an integer with* $0\le b\le n$. *Then* V^b *is free as an* R*–module with basis*

$$\{\, v_{\mathfrak{st}} \mid \mathfrak{s}\in\operatorname{Std}_{b,n-b}(\lambda) \text{ and } \mathfrak{t}\in\operatorname{Std}(\lambda) \text{ for some } \lambda\in\Lambda_b^+ \,\}$$

and $\ker\theta_b$ *is free as an* R*–module with basis*

$$\{\, m_{\mathfrak{uv}} \mid \mathfrak{u}\in\operatorname{Std}_b(\mu) \text{ and } \mathfrak{v}\in\operatorname{Std}(\mu) \text{ for some } \mu\in\overline{\Lambda}_b{}^+ \,\}.$$

Proof The homomorphism θ_b is surjective, so by Lemma 3.9 V^b is spanned by the elements $\theta_b(m_{\mathfrak{uv}})$, where $\mathfrak{u}\in\operatorname{Std}_b(\mu)$, $\mathfrak{v}\in\operatorname{Std}(\mu)$ and $\mu\in\Lambda_b^+\cup\overline{\Lambda}_b{}^+$. Furthermore, $\theta_b(m_{\mathfrak{uv}})=0$ whenever $\mu\in\overline{\Lambda}_b{}^+$, by Lemma 3.10. Finally, the elements $\{\, v_{\mathfrak{st}}=\theta_b(m_{\mathfrak{st}}) \mid \mathfrak{s}\in\operatorname{Std}_{b,n-b}(\lambda), \mathfrak{t}\in\operatorname{Std}(\lambda) \text{ and } \lambda\in\Lambda_b^+\,\}$ are linearly independent (and non–zero) by Lemma 3.11(ii). Combing these three statements proves the Theorem. □

Let $\overline{N}^b=\bigcap_{\lambda\in\Lambda_b^+}\overline{N}^\lambda=\sum_{\mu\in\overline{\Lambda}_b{}^+}N^\mu$. Then $\overline{N}^b$ is a two–sided ideal in $\mathscr{H}$ and it is free as an R–module with basis $\{\, m_{\mathfrak{uv}} \mid \mathfrak{u},\mathfrak{v}\in\operatorname{Std}(\mu) \text{ for some } \mu\in\overline{\Lambda}_b{}^+\,\}$.

3.14 Corollary *Suppose that* $0 \le b \le n$. *Then* $\ker\theta_b = M^{\omega_b} \cap \overline{N}^b$.

3.15 Remark The proof of Theorem 3.13 relies on the assumption that $f_s(q,\mathbf{Q})$ is an invertible element of R. Without assuming that $f_s(q,\mathbf{Q})$ is invertible it is possible to prove via a brute force calculation that V^b is free as an R–module with basis

$$\left\{\, v_b L_1^{d_1}\dots L_n^{d_n} T_w \;\middle|\; \begin{array}{c} w \in \mathfrak{S}_n,\ 0 \le d_i < s \text{ for } 1 \le i \le b \\ \text{and } 0 \le d_i < r-s \text{ for } b < i \le n \end{array} \right\}.$$

(A straightforward argument using the Robinson–Schensted correspondence verifies that the rank of V^b is the same in both cases.) That this set is a basis of V^b can also be deduced from Proposition 4.8 below; however, this argument requires that $f_s(q,\mathbf{Q})$ be invertible in R.

In [**16**, Corollary 4.15] Specht filtrations of the right ideals M^μ of $\mathscr{H}$ were constructed as a consequence of (2.10); we now refine the filtration of M^{ω_b} to give Specht filtrations of the modules V^b and $\ker\theta_b$.

3.16 Theorem *Suppose that* $f_s(q,\mathbf{Q})$ *is invertible in* R *and let* b *be an integer with* $0 \le b \le n$.

(i) *There is a filtration* $V^b = V_1 \supset V_2 \supset \dots \supset V_k \supset V_{k+1} = 0$ *of* V^b *such that for each* $1 \le i \le k$ *there exists a multipartition* $\lambda_i \in \Lambda_b^+$ *with* $V_i/V_{i+1} \cong S^{\lambda_i}$. *Moreover, for each* $\lambda \in \Lambda_b^+$ *the number of* i *with* $\lambda_i = \lambda$ *is* $|\operatorname{Std}_{b,n-b}(\lambda)|$.

(ii) *There is a filtration* $\ker\theta_b = K_1 \supset K_2 \supset \dots \supset K_l \supset K_{l+1} = 0$ *of* $\ker\theta_b$ *such that for each* $1 \le i \le l$ *there exists a multipartition* $\mu_i \in \overline{\Lambda}_b{}^+$ *with* $K_i/K_{i+1} \cong S^{\mu_i}$. *Moreover, for each* $\mu \in \overline{\Lambda}_b{}^+$ *the number of* i *with* $\mu_i = \mu$ *is* $|\operatorname{Std}_b(\mu)|$.

Proof We recall the construction from [**16**, Cor. 4.15], using Lemma 3.9 to adapt the notation. Let $\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_N$ be the tableaux in $\bigcup_{\lambda \in \Lambda_b^+ \cup \overline{\Lambda}_b{}^+} \operatorname{Std}_b(\lambda)$, ordered so that $j > i$ whenever $\lambda_i \rhd \lambda_j$; here we set $\lambda_i = \operatorname{Shape}(\mathfrak{s}_i)$ for all i. For $i = 1, 2, \dots, N$ let M_i be the R–submodule of M^{ω_b} with basis $\{\, m_{\mathfrak{s}_j\mathfrak{t}} \mid j \ge i \text{ and } \mathfrak{t} \in \operatorname{Std}(\lambda_j) \,\}$. Then the proof of [**16**, Cor. 4.15] shows that each M_i is an $\mathscr{H}$–submodule of M^{ω_b} and that $M_i/M_{i+1} \cong S^{\lambda_i}$ for all i.

Choose k to be maximal such that $\lambda_k \in \Lambda_b^+$; then $M_{k+1} = \ker\theta_b$ and $\lambda_i \in \overline{\Lambda}_b{}^+$ if and only if $i > k$. By Corollary 3.14 we can set $V_i = M_i/\ker\theta_b$, for $i = 1, 2, \dots, k{+}1$, to obtain a filtration of V^b with the required properties. Similarly, setting $K_j = M_{k+j}$, for $j = 1, 2, \dots, N{-}k$, gives the promised filtration of $\ker\theta_b$. □

Recall from before Corollary 2.8 that the content $\operatorname{cont}(\lambda)$ of λ is the sequence $(c_r)_{r\in R}$, where $c_r = \#\{\, x \in [\lambda] \mid \operatorname{res}(x) = r \,\}$ for all $r \in R$. Importantly, two Specht modules belong to the same block only if the corresponding multipartitions have the same content by Corollary 2.8.

3.17 Corollary *Suppose that $f_s(q,\mathbf{Q})$ is invertible and let b and c be distinct integers with $0 \le b, c \le n$. Then* $\operatorname{Hom}_{\mathscr{H}}(V^b, V^c) = 0$.

Proof Let $\mathcal{R}_s = \{\, q^d Q_i \mid -n < d < n \text{ and } 1 \le i \le s \,\}$; then $\mathcal{R}_s$ is the complete set of possible residues $\operatorname{res}(x)$, where x runs through the nodes in the first s–components of the diagram of any multipartition of n. In addition, since $f_s(q,\mathbf{Q})$ is invertible, if y is a node appearing in one of the last $r-s$ components of some multipartition then $\operatorname{res}(y) \notin \mathcal{R}_s$; consequently, if λ is a multipartition and $\operatorname{cont}(\lambda) = (c_r)_{r\in R}$ then $\lambda \in \Lambda_b^+$ if and only if $b = \sum_{r\in\mathcal{R}_s} c_r$. Therefore, if $\lambda \in \Lambda_b^+$ and $\mu \in \Lambda_c^+$ then $\operatorname{cont}(\lambda) \ne \operatorname{cont}(\mu)$ — note that by assumption $b \ne c$.

Now consider $\operatorname{Hom}_{\mathscr{H}}(V^b, V^c)$. By Theorem 3.16, V^b has a Specht filtration indexed by multipartitions in Λ_b^+ and V^c has a Specht filtration indexed by the multipartitions in Λ_c^+. Therefore, the simple composition factors of V^b and V^c belong to different blocks by the last paragraph and Corollary 2.8; hence, $\operatorname{Hom}_{\mathscr{H}}(V^b, V^c) = 0$ by Schur's lemma. □

Notice that $\overline{\Lambda}_b{}^+ = \bigcup_{c=b+1}^{n} \Lambda_c^+$; therefore, by a similar argument, again using Theorem 3.16 and Corollary 2.8, the composition factors of V^b and $\ker\theta_b$ belong to different blocks. Hence, we also have the following.

3.18 Corollary *Suppose $f_s(q,\mathbf{Q})$ is invertible in R and that $0 \le b \le n$. Then the composition factors of V^b and $\ker\theta_b$ belong to different blocks of $\mathscr{H}$; consequently, $M^{\omega_b} \cong V^b \oplus \ker\theta_b$ and $\operatorname{End}_{\mathscr{H}}(M^{\omega_b}) \cong \operatorname{End}_{\mathscr{H}}(V^b) \oplus \operatorname{End}_{\mathscr{H}}(\ker\theta_b)$.*

In fact, this allows us to determine a basis of $\operatorname{End}_{\mathscr{H}}(V^b)$. Given two standard λ–tableaux $\mathfrak{s}$ and $\mathfrak{t}$ in $\operatorname{Std}_{b,n-b}(\lambda)$ let $\theta_{\mathfrak{s}\mathfrak{t}} \colon V^b \longrightarrow V^b$ be the R–linear map given by $\theta_{\mathfrak{s}\mathfrak{t}}(v_b h) = v_{\mathfrak{s}\mathfrak{t}} h$ for all $h \in \mathscr{H}$; then $\theta_{\mathfrak{s}\mathfrak{t}}$ is an R–module homomorphism. *A priori* there is no reason to expect that $\theta_{\mathfrak{s}\mathfrak{t}}$ is even well defined; nevertheless, it is and these elements give a basis of $\operatorname{End}_{\mathscr{H}}(V^b)$.

3.19 Theorem *Suppose $f_s(q,\mathbf{Q})$ is invertible in R and let $0 \le b \le n$. Then $\operatorname{End}_{\mathscr{H}}(V^b)$ is free as an R–module with basis*

$$\{\, \theta_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}_{b,n-b}(\lambda) \text{ for some } \lambda \in \Lambda_b^+ \,\}.$$

Proof A basis of $\operatorname{End}_{\mathscr{H}}(M^{\omega_b})$ is given by (2.11); in light of Lemma 3.9, we see that $\operatorname{End}_{\mathscr{H}}(M^{\omega_b})$ has as basis the maps

$$\{\, \varphi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}_b(\lambda) \text{ for some } \lambda \in \Lambda_b^+ \cup \overline{\Lambda}_b{}^+ \,\},$$

where $\varphi_{\mathfrak{s}\mathfrak{t}}$ is given by $\varphi_{\mathfrak{s}\mathfrak{t}}(m_{\omega_b} h) = m_{\mathfrak{s}\mathfrak{t}} h$ for all $h \in \mathscr{H}$.

By Corollary 3.18 the homomorphism $\theta_b \colon M^{\omega_b} \longrightarrow V^b$ splits, so θ_b has a right inverse; we abuse notation and write θ_b^{-1} for this one sided inverse. Let φ be any map in $\operatorname{End}_{\mathscr{H}}(M^{\omega_b})$; then $\theta_b \varphi \theta_b^{-1}$ belongs to $\operatorname{End}_{\mathscr{H}}(V^b)$ and every homomorphism in $\operatorname{End}_{\mathscr{H}}(V^b)$ is of this form. Now, $\theta_b(m_{\omega_b}) = v_b$; so there exists an $h_b \in \ker\theta_b$ such that $\theta_b^{-1}(v_b h) = (m_{\omega_b} + h_b)h$ for all $h \in \mathscr{H}$. Observe that $\varphi_{\mathfrak{s}\mathfrak{t}}(h_b) \in \ker\theta_b$ since $\ker\theta_b$ are V^b are in different blocks by Corollary 3.18. Therefore, by Theorem 3.13,

$$\theta_b\varphi_{\mathfrak{s}\mathfrak{t}}\theta_b^{-1}(v_b h) = \theta_b\varphi_{\mathfrak{s}\mathfrak{t}}(m_{\omega_b} + h_b)h = \theta_b(m_{\mathfrak{s}\mathfrak{t}})h = \begin{cases} v_{\mathfrak{s}\mathfrak{t}}h, & \text{if } \lambda \in \Lambda_b^+, \\ 0, & \text{if } \lambda \in \overline{\Lambda}_b{}^+. \end{cases}$$

Consequently, $\theta_b\varphi_{\mathfrak{s}\mathfrak{t}}\theta_b^{-1} = \theta_{\mathfrak{s}\mathfrak{t}}$ if $\lambda \in \Lambda_b^+$ and $\theta_b\varphi_{\mathfrak{s}\mathfrak{t}}\theta_b^{-1} = 0$ if $\lambda \in \overline{\Lambda}_b{}^+$; in particular, $\theta_{\mathfrak{s}\mathfrak{t}} \in \operatorname{End}_{\mathscr{H}}(V^b)$ whenever $\mathfrak{s}$ and $\mathfrak{t}$ belong to $\operatorname{Std}_{b,n-b}(\lambda)$. As the elements $\{\, v_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \operatorname{Std}_{b,n-b}(\lambda) \text{ for some } \lambda \in \Lambda_b^+ \,\}$ are linearly independent, so are the corresponding homomorphisms $\{\theta_{\mathfrak{s}\mathfrak{t}}\}$. The Theorem follows. □

The argument also shows that a basis of $\operatorname{End}_{\mathscr{H}}(\ker\theta_b)$ is given by restricting the maps $\{\, \varphi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \operatorname{Std}_b(\mu) \text{ for some } \mu \in \overline{\Lambda}_b{}^+ \,\}$ to $\ker\theta_b$.

The final property of the modules V^b that we need is that they are projective $\mathscr{H}$–modules. To prove this let $\mathscr{D}_{(b,n-b)}$ be the set of distinguished right coset representatives of $\mathfrak{S}_{(b,n-b)}$ in $\mathfrak{S}_n$; see, for example, [**24**, Prop. 3.3].

3.20 Theorem *Suppose that $f_s(q,\mathbf{Q})$ is invertible. Then* $\mathscr{H} \cong \bigoplus_{b=0}^{n} \binom{n}{b} V^b$.

Proof Suppose that $0 \le b \le n$. As in the proof of Theorem 3.19 let θ_b^{-1} be a right inverse to θ_b and set $V_b' = \theta_b^{-1}(V^b)$; then $V_b' \cong V^b$ is a submodule of M^{ω_b}. For each multipartition $\lambda \in \Lambda_b^+$ and tableaux $\mathfrak{s} \in \operatorname{Std}_{b,n-b}(\lambda)$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$ let $v'_{\mathfrak{s}\mathfrak{t}} = \theta_b^{-1}(v_{\mathfrak{s}\mathfrak{t}})$; then $v'_{\mathfrak{s}\mathfrak{t}} \in V_b'$ and $v'_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{s}\mathfrak{t}} + h_{\mathfrak{s}\mathfrak{t}}$ for some $h_{\mathfrak{s}\mathfrak{t}} \in \ker\theta_b$. Moreover, the set of these elements is a basis of V_b'.

We claim that $\mathscr{H} = \bigoplus_{b=0}^n \bigoplus_{w\in\mathscr{D}_{(n-b,b)}} T_w^* V_b'$; since $[\mathfrak{S}_n : \mathfrak{S}_{(b,n-b)}] = \binom{n}{b}$ this will establish the Theorem. Note that $T_w^* V_b' \cong V^b$ as right $\mathscr{H}$–modules.

If $w \in \mathscr{D}_{(b,n-b)}$ and $\mathfrak{s} \in \operatorname{Std}_{b,n-b}(\lambda)$ then $\mathfrak{s}_w = \mathfrak{s}w$ is again a standard λ–tableau, $d(\mathfrak{s}_w) = d(\mathfrak{s})w$ and $\ell(d(\mathfrak{s}_w)) = \ell(d(\mathfrak{s})) + \ell(w)$. Therefore, $T_w^* V_b'$ has as basis the elements

$$T_w^* v'_{\mathfrak{s}\mathfrak{t}} = T_w^*\big(m_{\mathfrak{s}\mathfrak{t}} + h_{\mathfrak{s}\mathfrak{t}}\big) = m_{\mathfrak{s}_w\mathfrak{t}} + T_w^* h_{\mathfrak{s}\mathfrak{t}},$$

where $\mathfrak{s},\mathfrak{t} \in \operatorname{Std}_{b,n-b}(\lambda)$ for some $\lambda \in \Lambda_b^+$. Notice that $\ker\theta_b \subseteq \overline{N}^b$ by Corollary 3.14 and that $\overline{N}^b$ is a two–sided ideal of $\mathscr{H}$; so $T_w^* h_{\mathfrak{s}\mathfrak{t}} \in \overline{N}^b$.

Now, if $\lambda \in \Lambda_b^+$ then $\operatorname{Std}(\lambda) = \coprod_{w\in\mathscr{D}_{(b,n-b)}} \operatorname{Std}_{b,n-b}(\lambda)w$. Therefore, the elements

$$\bigcup_{b=0}^{n} \{\, T_w^* v'_{\mathfrak{s}\mathfrak{t}} \mid w \in \mathscr{D}_{(b,n-b)}, \mathfrak{s} \in \operatorname{Std}_{b,n-b}(\lambda) \text{ and } \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some } \lambda \in \Lambda_b^+ \,\}$$

are linearly independent; moreover, by (2.3) and the above remarks, this set is also a basis of $\mathscr{H}$. Consequently, $\mathscr{H} = \bigoplus_{b=0}^n \bigoplus_{w\in\mathscr{D}_{(b,n-b)}} T_w^* V_b'$, proving our claim and hence the Theorem. □

A similar argument shows that $M^{\omega_a} \cong \bigoplus_{b=a}^n \binom{n-a}{b} V^b$ whenever $0 \le a \le n$; the Theorem is the case $a = 0$ since $m_{\omega_0} = 1$.

Recall that a progenerator, or projective generator, for $\mathscr{H}$ is a projective $\mathscr{H}$–module which contains every principal indecomposable $\mathscr{H}$–module as a direct summand. Every direct summand of $\mathscr{H}$ is automatically projective and every principal indecomposable $\mathscr{H}$–module is a direct summand of $\mathscr{H}$; hence, Theorem 3.20 yields the following.

3.21 Corollary *Suppose that $f_s(q, \mathbf{Q})$ is invertible. Then*
(i) *V^b is a projective $\mathscr{H}$–module for $b = 0, 1, \dots, n$; and,*
(ii) *$V = \bigoplus_{b=0}^{n} V^b$ is a progenerator for $\mathscr{H}$.*

4 The Morita equivalence

We are almost ready to prove our main results; in fact, we have already done most of the hard work. In Corollary 3.21(ii) we showed that $V = \bigoplus_{b=0}^n V^b$ is a progenerator of $\mathscr{H}$; hence, it follows that $\mathscr{H}$ is Morita equivalent to $\operatorname{End}_{\mathscr{H}}(V)$. To prove Theorem 1.1 we show that $\operatorname{End}_{\mathscr{H}}(V)$ is isomorphic to a direct sum of tensor products of smaller Ariki–Koike algebras.

4.1 Proposition *Suppose that $f_s(q, \mathbf{Q})$ is invertible. Then $\mathscr{H}$ is Morita equivalent to $\bigoplus_{b=0}^n \operatorname{End}_{\mathscr{H}}(V^b)$.*

Proof Let $V = \bigoplus_{b=0}^n V^b$. As remarked above, it follows from Corollary 3.21(ii) that $\mathscr{H}$ is Morita equivalent to $\operatorname{End}_{\mathscr{H}}(V)$; see [**7**, Lemma 2.2.3]. However, by Corollary 3.17,

$$\operatorname{End}_{\mathscr{H}}(V) = \bigoplus_{0 \le b,c \le n} \operatorname{Hom}_{\mathscr{H}}(V^b, V^c) \cong \bigoplus_{b=0}^{n} \operatorname{End}_{\mathscr{H}}(V^b),$$

giving the result. □

For $b = 0, 1 \dots, n$ let $\mathscr{H}_b \otimes \mathscr{H}_{n-b} = \mathscr{H}_{q,\mathbf{Q}_1}(b) \otimes \mathscr{H}_{q,\mathbf{Q}_2}(n-b)$, where $\mathbf{Q}_1 = (Q_1, \dots, Q_s)$ and $\mathbf{Q}_2 = (Q_{s+1}, \dots, Q_r)$. We have already computed $\operatorname{End}_{\mathscr{H}}(V^b)$ in Theorem 3.19; in order to prove Theorem 1.6, and hence our main result, we will use this result to show that $\operatorname{End}_{\mathscr{H}}(V^b) \cong \mathscr{H}_b \otimes \mathscr{H}_{n-b}$.

The subalgebra $\mathscr{H}(\mathfrak{S}_b) \otimes \mathscr{H}(\mathfrak{S}_{n-b})$ of $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ is isomorphic to the subalgebra $\mathscr{H}(\mathfrak{S}_{(n-b,b)})$ of $\mathscr{H}$ spanned by $\{\, T_w \mid w \in \mathfrak{S}_{(n-b,b)} \,\}$ — note that $\mathscr{H}(\mathfrak{S}_{(b,n-b)})$ and $\mathscr{H}(\mathfrak{S}_{(n-b,b)})$ are isomorphic algebras, the reason for introducing the twist is that $\mathscr{H}(\mathfrak{S}_{(n-b,b)})v_b = v_b\mathscr{H}(\mathfrak{S}_{(b,n-b)})$ by Proposition 3.4. In general $\mathscr{H}$ has no subalgebra isomorphic to $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$; however, $\mathscr{H}$ does have an R–submodule isomorphic to $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ and this we can exploit. Let $\mathscr{H}_{n-b,b}$ be the R–submodule of $\mathscr{H}$ spanned by the elements

$$\left\{ L_1^{d_1} \dots L_n^{d_n} T_w \;\middle|\; \begin{array}{c} w \in \mathfrak{S}_{(n-b,b)},\ 0 \le d_i < r-s \text{ for } 1 \le i \le n-b \\ \text{and } 0 \le d_i < s \text{ for } n-b < i \le n \end{array} \right\}.$$

We emphasize that typically $\mathscr{H}_{n-b,b}$ is not a subalgebra of $\mathscr{H}$; however, $\mathscr{H}_{n-b,b}$ does generate the subalgebra H_b of Corollary 3.5.

4.2 *Suppose that $0 \le b \le n$. Then $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ and $\mathscr{H}_{n-b,b}$ are isomorphic as R–modules via the R–linear map $\Theta_b : \mathscr{H}_b \otimes \mathscr{H}_{n-b} \longrightarrow \mathscr{H}_{n-b,b}$ determined by*

$$L_1^{d_1} \dots L_b^{d_b} T_x \otimes L_1^{e_1} \dots L_{n-b}^{e_{n-b}} T_y \longmapsto (L_1^{e_1} \dots L_{n-b}^{e_{n-b}} T_y)(L_{n-b+1}^{d_1} \dots L_n^{d_b} T_{x'}),$$

where $x' = w_{n-b,b} x w_{b,n-b} = w_{b,n-b}^{-1} x w_{b,n-b}$; here $x \in \mathfrak{S}_b$, $y \in \mathfrak{S}_{n-b}$, $0 \le d_i < s$ for $i = 1, \dots, b$ and $0 \le e_j < r-s$ for $j = 1, \dots, n-b$.

Maintaining the notation of (4.2) notice that

$$(L_1^{e_1} \dots L_{n-b}^{e_{n-b}} T_y)(L_{n-b+1}^{d_1} \dots L_n^{d_b} T_{x'}) = L_1^{e_1} \dots L_{n-b}^{e_{n-b}} L_{n-b+1}^{d_1} \dots L_n^{d_b} T_{x'y}$$

by (2.1); in particular, the map $\mathfrak{S}_b \times \mathfrak{S}_{n-b} \longrightarrow \mathfrak{S}_{(n-b,b)} : (x,y) \longmapsto x'y = yx'$ is an isomorphism of groups. Hereafter, we identify elements of the R–modules $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ and $\mathscr{H}_{n-b,b}$ via the map Θ_b.

4.3 Lemma *Suppose that $0 \le b \le n$. Then V^b becomes a left $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$–module via $hv = \Theta_b(h)v$ for all $h \in \mathscr{H}_b \otimes \mathscr{H}_{n-b}$ and $v \in V^b$. Consequently, $\Theta_b(h_1h_2)v = \Theta_b(h_1)\Theta_b(h_2)v$ for all $h_1, h_2 \in \mathscr{H}_b \otimes \mathscr{H}_{n-b}$ and all $v \in V^b$.*

Proof It is enough to check that the relations in $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ are preserved. As an algebra $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ is generated by the two sets of commuting elements $T_i \otimes 1$ and $1 \otimes T_j$, where $0 \le i < b$ and $0 \le j < n-b$. Now

$$\Theta_b(T_i \otimes 1) = \begin{cases} L_{n-b+1}, & \text{if } i = 0, \\ T_{n-b+i}, & \text{otherwise,} \end{cases} \quad \text{and} \quad \Theta_b(1 \otimes T_j) = \begin{cases} L_1, & \text{if } j = 0, \\ T_j, & \text{otherwise.} \end{cases}$$

Therefore, the relations in $\mathscr{H}$ and (2.1)(ii) ensure that all of the relations in $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ are satisfied except possibly that $(T_0 \otimes 1 - Q_1) \dots (T_0 \otimes 1 - Q_s)$ and $(1 \otimes T_0 - Q_{s+1}) \dots (1 \otimes T_0 - Q_r)$ must both act as zero on V^b. However, this is precisely the content of parts (ii) and (i), respectively, of Corollary 3.6. (Notice that when Θ_b is applied to the braid relation $T_0T_1T_0T_1 = T_1T_0T_1T_0$ in $\mathscr{H}_b$ it becomes $qL_{n-b+1}L_{n-b+2} = qL_{n-b+2}L_{n-b+1}$ in $\mathscr{H}$; this holds by virtue of (2.1)(ii).) $\square$

Therefore, V^b is an $(\mathscr{H}_b \otimes \mathscr{H}_{n-b}, \mathscr{H})$–bimodule. Evidently, the left action of $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ on V^b commutes with the right action of $\mathscr{H}$ so we have a homomorphism from $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ into $\mathrm{End}_{\mathscr{H}}(V^b)$. Furthermore, Lemma 4.3

implies that for all $h_1, h_2 \in \mathscr{H}_b \otimes \mathscr{H}_{n-b}$ there exists some $h_3 \in \mathscr{H}$ such that $\Theta_b(h_1h_2) = \Theta_b(h_1)\Theta_b(h_2) + h_3$ and $h_3 \in \mathsf{L}_{H_b}(V^b)$; *cf.* Corollary 3.5.

Shortly we will see that $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ acts faithfully on V^b; before we can show this we need some more notation. Recall that $\Lambda^+ = \Lambda^+(n:r)$ and, more generally, that $\Lambda^+(b:s)$ is the set of multipartitions of b with s–components. There is an evident bijection $\Lambda^+(b:s)\times\Lambda^+(n-b:r-s) \longrightarrow \Lambda_b^+$ which maps the pair $\sigma = (\sigma^{(1)},\dots,\sigma^{(s)}) \in \Lambda^+(b:s)$ and $\tau = (\tau^{(1)},\dots,\tau^{(r-s)}) \in \Lambda^+(n-b:r-s)$ to the multipartition $(\sigma,\tau) = (\sigma^{(1)},\dots,\sigma^{(s)},\tau^{(1)},\dots,\tau^{(r-s)}) \in \Lambda_b^+$.

4.4 Lemma *Suppose that $0 \le b \le n$ and that $\lambda \in \Lambda_b^+$. Then $\lambda = (\sigma,\tau)$ for unique multipartitions $\sigma \in \Lambda^+(b:s)$ and $\tau \in \Lambda^+(n-b:r-s)$; moreover, we have $\theta_b(u_\lambda^+) = \Theta_b(u_\sigma^+ \otimes u_\tau^+)v_b$.*

Proof The uniqueness of σ and τ follows from the remarks above. For the remaining statement first recall that $u_{\alpha,t} = \prod_{k=1}^{\alpha}(L_k - Q_t)$ for any α and t. Let $\alpha_t = |\lambda^{(1)}| + \dots + |\lambda^{(t-1)}|$, for $1 \le t \le r$, and let $\beta_t = \alpha_{t+r-s} - b$ for $t = 1,\dots,r-s$. Then $u_\lambda^+ = u_{\alpha_1,1}\dots u_{\alpha_r,r}$ and, abusing notation, $u_\sigma^+ \otimes u_\tau^+ = u_{\alpha_1,1}\dots u_{\alpha_s,s} \otimes u_{\beta_1,s+1}\dots, u_{\beta_{r-s},r} \in \mathscr{H}_b \otimes \mathscr{H}_{n-b}$.

Now, by Proposition 3.4(iv), if $1 \le t \le s$ then

$$\Theta_b(u_{\alpha_t,t}\otimes 1)v_b = \Big(\prod_{k=n-b+1}^{n-b+\alpha_t}(L_k - Q_t)\Big)v_b = v_b u_{\alpha_t,t};$$

so, $\Theta_b(u_\sigma^+\otimes 1)v_b = v_b u_{\alpha_1,1}\dots u_{\alpha_s,s}$. On the other hand, by Proposition 3.4(iii),

$$\Theta_b(1\otimes u_{\beta_t,t})v_b = \Big(\prod_{k=1}^{\beta_t}(L_k - Q_t)\Big)v_b = v_b\prod_{k=b+1}^{b+\beta_t}(L_k - Q_t),$$

for $t = s+1,\dots,r$. Now, $u_{\alpha_t,t} = \prod_{k=1}^{b+\beta_t}(L_k - Q_t)$ for $t = s+1,\dots,r$; therefore,

$$\begin{aligned}\Theta_b(1\otimes u_\tau^+)v_b &= v_b\prod_{t=s+1}^{r}\prod_{k=b+1}^{b+\beta_t}(L_k - Q_t)\\ &= u_{n-b}^- T_{w_{n-b,b}} u_b^+ \prod_{t=s+1}^{r}\prod_{k=b+1}^{b+\beta_t}(L_k - Q_t)\\ &= u_{n-b}^- T_{w_{n-b,b}} u_{\alpha_{s+1},s+1}\dots u_{\alpha_r,r}.\end{aligned}$$

Hence, $\Theta_b(u_\sigma^+\otimes u_\tau^+)v_b = u_{n-b}^- T_{w_{n-b,b}} u_{\alpha_1,1}\dots u_{\alpha_r,r} = \theta_b(u_\lambda^+)$ as claimed. □

We want to extend this result to the standard basis of $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$. Suppose that $\lambda \in \Lambda_b^+$ and, as above, write $\lambda = (\sigma,\tau)$ for multipartitions $\sigma \in \Lambda^+(b:s)$ and $\tau \in \Lambda^+(n-b:r-s)$. Given standard tableaux $\mathfrak{s}_1 = (\mathfrak{s}_1^{(1)},\dots,\mathfrak{s}_1^{(s)}) \in \operatorname{Std}(\sigma)$ and $\mathfrak{s}_2 = (\mathfrak{s}_2^{(1)},\dots,\mathfrak{s}_2^{(r-s)}) \in \operatorname{Std}(\tau)$ let $(\mathfrak{s}_1,\mathfrak{s}_2')$ be the λ–tableau

$$(\mathfrak{s}_1,\mathfrak{s}_2') = (\mathfrak{s}_1^{(1)},\dots,\mathfrak{s}_1^{(s)},\mathfrak{s}_2^{(1)}w_{n-b,b},\dots,\mathfrak{s}_2^{(r-s)}w_{n-b,b});$$

then $(\mathfrak{s}_1, \mathfrak{s}_2') \in \operatorname{Std}_{b,n-b}(\lambda)$ and a straightforward calculation reveals the following.

4.5 Lemma *Suppose that $\lambda = (\sigma, \tau) \in \Lambda_b^+$ as above. Then the map*

$$\operatorname{Std}(\sigma) \times \operatorname{Std}(\tau) \longrightarrow \operatorname{Std}_{b,n-b}(\lambda) : (\mathfrak{s}_1, \mathfrak{s}_2) \longmapsto \mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2')$$

is a bijection. Furthermore, $d(\mathfrak{s}) = d(\mathfrak{s}_1) w_{n-b,b}^{-1} d(\mathfrak{s}_2) w_{n-b,b} \in \mathfrak{S}_{(b,n-b)}$ and $\ell(d(\mathfrak{s})) = \ell(d(\mathfrak{s}_1)) + \ell(d(\mathfrak{s}_2))$.

We can now give the connection between θ_b and Θ_b.

4.6 Lemma *Suppose $\lambda = (\sigma, \tau) \in \Lambda_b^+$ as above. Let $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}_{b,n-b}(\lambda)$ and write $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2')$ and $\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2')$ as in Lemma 4.5. Then*

$$\theta_b(m_{\mathfrak{s}\mathfrak{t}}) = \Theta_b(m_{\mathfrak{s}_1\mathfrak{t}_1} \otimes m_{\mathfrak{s}_2\mathfrak{t}_2}) v_b.$$

Proof By Proposition 3.4, $\mathscr{H}(\mathfrak{S}_{(n-b,b)}) v_b = v_b \mathscr{H}(\mathfrak{S}_{(b,n-b)})$; therefore,

$$\Theta_b(x_\sigma \otimes x_\tau) v_b = x_{(\tau,\sigma)} v_b = v_b x_{(\sigma,\tau)} = v_b x_\lambda.$$

Furthermore, $\Theta_b(u_\sigma^+ \otimes u_\tau^+) v_b = \theta_b(u_\lambda^+)$ by Lemma 4.4; therefore, using Lemma 4.3 to combine these two equations shows that

$$\begin{aligned}\Theta_b(m_\sigma \otimes m_\tau) v_b &= \Theta_b(x_\sigma u_\sigma^+ \otimes x_\tau u_\tau^+) v_b = \Theta_b(x_\sigma \otimes x_\tau) \Theta_b(u_\sigma^+ \otimes u_\tau^+) v_b \\ &= \Theta_b(x_\sigma \otimes x_\tau) v_b u_\lambda^+ = v_b x_\lambda u_\lambda^+ = v_b m_\lambda.\end{aligned}$$

Finally, note that if $x \in \mathfrak{S}_b$ and $y \in \mathfrak{S}_{n-b}$ then, by (4.2) and Proposition 3.4,

$$\Theta_b(T_x \otimes T_y) v_b = T_{w_{b,n-b}^{-1} x w_{b,n-b}} T_y v_b = v_b T_x T_{w_{n-b,b}^{-1} y w_{n-b,b}}.$$

Since $m_{\mathfrak{s}_1\mathfrak{t}_1} \otimes m_{\mathfrak{s}_2\mathfrak{t}_2} = (T_{d(\mathfrak{s}_1)}^* \otimes T_{d(\mathfrak{t}_1)})(m_\sigma \otimes m_\tau)(T_{d(\mathfrak{s}_2)}^* \otimes T_{d(\mathfrak{t}_2)})$, another application of Lemma 4.3, together with Lemma 4.5, now completes the proof. □

We have finally reached the summit.

4.7 Theorem *Suppose that $f_s(q, \mathbf{Q})$ is invertible in R and let b be an integer with $0 \le b \le n$. Then*

(i) *V^b is a faithful left $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$–module;*

(ii) *$\operatorname{End}_{\mathscr{H}}(V^b) \cong \mathscr{H}_b \otimes \mathscr{H}_{n-b}$; and,*

(iii) *$\mathscr{H}$ is Morita equivalent to $\mathscr{H}_s = \bigoplus_{b=0}^{n} \mathscr{H}_b \otimes \mathscr{H}_{n-b}$.*

Proof By Lemma 4.3 $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$ acts on V^b by left multiplication; this action is faithful because, by Lemma 4.6 and Theorem 3.13, the elements of the standard basis of $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$ map v_b to linearly independent elements in V^b. Further, because the action is faithful, $\mathrm{End}_{\mathscr{H}}(V^b)\cong\mathscr{H}_b\otimes\mathscr{H}_{n-b}$ by Theorem 3.19 and Lemma 4.6. Finally, part (iii) follows from part (ii) and Proposition 4.1. □

This proves Theorem 1.6 and hence the main result of this paper. The remainder of this section examines this Morita equivalence more closely.

For an algebra A let $\mathbf{Mod}_A$ be the category of (finite dimensional) right A–modules. Then we have shown that there exist functors (in fact, category equivalences),

$$\mathrm{H_s}:\mathbf{Mod}_{\mathscr{H}}\longrightarrow\mathbf{Mod}_{\mathscr{H}_s}\quad\text{and}\quad\hat{\mathrm{H}}_{\mathrm{s}}:\mathbf{Mod}_{\mathscr{H}_s}\longrightarrow\mathbf{Mod}_{\mathscr{H}}.$$

These functors are described in terms of the $(\mathscr{H}_s,\mathscr{H})$–bimodule $V=\bigoplus_{b=0}^n V^b$; explicitly, $\mathrm{H_s(M)}=\mathrm{Hom}_{\mathscr{H}}(\mathrm{V},\mathrm{M})$ and $\hat{\mathrm{H}}_s(X)=X\otimes_{\mathscr{H}_s}V$. (Here we consider V as a left $\mathscr{H}_s$–module by specifying that $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$ annihilates V^c when $b\neq c$.) For this, and other standard facts about Morita equivalences, see [**7**, Section 2.2].

By Theorem 3.20 we can write $\mathscr{H}=\bigoplus_{b=0}^n\mathscr{H}(b)$ where $\mathscr{H}(b)$ is the smallest two–sided ideal of $\mathscr{H}$ which contains V^b as a direct summand (for $0\le b\le n$). Therefore, $\mathbf{Mod}_{\mathscr{H}}=\bigoplus_{b=0}^n\mathbf{Mod}_{\mathscr{H}(b)}$. The modules in $\mathbf{Mod}_{\mathscr{H}(b)}$ are mapped into $\mathbf{Mod}_{\mathscr{H}_b\otimes\mathscr{H}_{n-b}}$ by $\mathrm{H_s}$, so we have subfunctors

$$\mathrm{H_{s,b}}:\mathbf{Mod}_{\mathscr{H}(\mathrm{b})}\longrightarrow\mathbf{Mod}_{\mathscr{H}_\mathrm{b}\otimes\mathscr{H}_{\mathrm{n-b}}}\quad\text{and}\quad\hat{\mathrm{H}}_{\mathrm{s,b}}:\mathbf{Mod}_{\mathscr{H}_\mathrm{b}\otimes\mathscr{H}_{\mathrm{n-b}}}\longrightarrow\mathbf{Mod}_{\mathscr{H}(\mathrm{b})}$$

given by $\mathrm{H_{s,b}(M)}=\mathrm{Hom}_{\mathscr{H}}(\mathrm{V^b},\mathrm{M})$ and $\hat{\mathrm{H}}_{s,b}(X)=X\otimes_{\mathscr{H}_b\otimes\mathscr{H}_{n-b}}V^b$. Each of these functors induces a Morita equivalence.

In general, V is not free as a left $\mathscr{H}_s$–module; however, as the next result shows, V^b is free as a left $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$–module. Recall that $\mathscr{D}_{(b,n-b)}$ is the set of distinguished right coset representatives of $\mathfrak{S}_{(b,n-b)}\cong\mathfrak{S}_b\times\mathfrak{S}_{n-b}$ in $\mathfrak{S}_n$. For each $w\in\mathscr{D}_{(b,n-b)}$ let V^b_w be the R–submodule $\Theta_b(\mathscr{H}_b\otimes\mathscr{H}_{n-b})v_bT_w$ of V^b.

4.8 Proposition *Suppose that $0\le b\le n$ and let $w\in\mathscr{D}_{(b,n-b)}$. Then V^b_d is a left $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$–submodule of V^b which is isomorphic to the left regular representation of $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$. A basis of V^b_d is given by*

$$\{\,v_{\mathfrak{s}\mathfrak{t}}\mid\mathfrak{s}\in\mathrm{Std}_{b,n-b}(\lambda)\text{ and }\mathfrak{t}\in\mathrm{Std}(\lambda)w\text{ for some }\lambda\in\Lambda^+_b\,\}.$$

Moreover, as a left $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$–module,

$$V^b=\bigoplus_{w\in\mathscr{D}_{(b,n-b)}}V^b_w.$$

Consequently, V^b is a free $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$–module of rank $\binom{n}{b}$.

Proof Applying Lemma 4.6 and Theorem 3.13 we see that V^b_w has basis $\{\,v_{\mathfrak{s}\mathfrak{t}}T_w\mid\mathfrak{s},\mathfrak{t}\in\mathrm{Std}_{b,n-b}(\lambda)\text{ for some }\lambda\in\Lambda^+_b\,\}$. Now, if $\mathfrak{t}\in\mathrm{Std}_{b,n-b}(\lambda)$ then

$d(\mathfrak{t}) \in \mathfrak{S}_{b,n-b}$ so that $\ell(d(\mathfrak{t})w) = \ell(d(\mathfrak{t})) = \ell(w)$ since w is a distinguished right coset representative of $\mathfrak{S}_{(b,n-b)}$ in $\mathfrak{S}_n$. Therefore, if $\mathfrak{s}$ and $\mathfrak{t}$ in $\operatorname{Std}_{b,n-b}(\lambda)$ then $m_{\mathfrak{st}}T_w = m_{\mathfrak{su}}$ where $\mathfrak{u} = \mathfrak{t}w \in \operatorname{Std}(\lambda)$; so, $v_{\mathfrak{st}}T_w = v_{\mathfrak{su}}$. Therefore, V_w^b has the required basis and $V^b = \bigoplus_{w\in\mathscr{D}_{(b,n-b)}} V_w^b$. Finally, V_w^b affords the left regular representation of $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ by Theorem 4.7(i). □

Suppose that $\{\, h_i \mid i \in I \,\}$ is a basis of $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$; then, by the Proposition,

$$\{\, \Theta_b(h_i)v_bT_w \mid i \in I \text{ and } w \in \mathscr{D}_{(b,n-b)} \,\}$$

is a basis of V^b. In particular, taking $\{\, h_i \mid i \in I \,\}$ to be the Ariki–Koike basis of $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ we obtain the basis of V^b mentioned in Remark 3.15.

In particular, since V^b is free as a left $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$–module by Proposition 4.8, tensoring with V^b is an exact functor. Hence, we have the following.

4.9 Corollary *Suppose that X is a right ideal of $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$. Then*

$$\hat{\mathrm{H}}_{s,b}(X) = \Theta_b(X)V^b = \bigoplus_{w\in\mathscr{D}_{(b,n-b)}} \Theta_b(X)v_bT_w,$$

where $\Theta_b(X)v_bT_w = \Theta_b(X)V_w^b \cong X$ as an R–module. In particular, if X is a free R–module of rank ℓ then $\hat{\mathrm{H}}_{s,b}(X)$ is a free R–module of rank $\ell\binom{n}{b}$.

More generally, if $0 \longrightarrow X \longrightarrow Y \longrightarrow Y/X \longrightarrow 0$ is an exact sequence of $\mathscr{H}_b\otimes\mathscr{H}_{n-b}$–modules then $\hat{\mathrm{H}}_{s,b}(Y/X) \cong \hat{\mathrm{H}}_{s,b}(Y)/\hat{\mathrm{H}}_{s,b}(X) \cong \Theta_b(Y)V^b/\Theta_b(X)V^b$.

We are now ready to describe how the Specht modules S^λ and simple $\mathscr{H}$–modules D^μ of $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ "multiply up" to give $\mathscr{H}$–modules.

4.10 Lemma *Suppose that R is a field, $0 \le b \le n$ and that $\lambda \in \Lambda_b^+$. Then $S^\lambda \in \mathbf{Mod}_{\mathscr{H}(b)}$; moreover, if $D^\lambda \neq (0)$ then $D^\lambda \in \mathbf{Mod}_{\mathscr{H}(b)}$.*

Proof First note that by [**18**, Theorem 3.7(ii)] all of the composition factors of S^λ belong to the same block; therefore, by Theorem 3.16(i) if $\lambda \in \Lambda_b^+$ then $S^\lambda \in \mathbf{Mod}_{\mathscr{H}(b)}$. Finally, if $D^\lambda \neq (0)$ then S^λ and D^λ belong to the same block, so D^λ also belongs to $\mathbf{Mod}_{\mathscr{H}(b)}$. □

Consequently, if $\lambda \in \Lambda_b^+$ then $\hat{\mathrm{H}}_s(S^\lambda) = \hat{\mathrm{H}}_{s,b}(S^\lambda)$ and $\hat{\mathrm{H}}_s(D^\lambda) = \hat{\mathrm{H}}_{s,b}(D^\lambda)$.

Recall that $d_{\lambda\mu} = [S^\lambda : D^\mu]$ is the decomposition multiplicity of the simple module D^μ in the Specht module S^λ.

4.11 Proposition *Suppose that R is a field and that $f_s(q,\mathbf{Q}) \neq 0$. Let $\lambda \in \Lambda_b^+$ and $\mu \in \Lambda_c^+$ be multipartitions of n and, as in Lemma 4.4, write $\lambda = (\sigma,\tau)$ and $\mu = (\alpha,\beta)$, where $\sigma \in \Lambda^+(b:s)$, $\tau \in \Lambda^+(n-b:r-s)$, $\alpha \in \Lambda^+(c:s)$ and $\beta \in \Lambda^+(n-c:r-s)$ (and $0 \le b,c \le n$). Then the following hold.*

(i) $S^\lambda \cong \hat{\mathrm{H}}_s(S^\sigma \otimes S^\tau) = \hat{\mathrm{H}}_{s,b}(S^\sigma \otimes S^\tau)$.

(ii) $D^\mu \cong \hat{\mathrm{H}}_s(D^\alpha \otimes D^\beta) = \hat{\mathrm{H}}_{s,c}(D^\alpha \otimes D^\beta)$. *Consequently,* $D^\mu \neq (0)$ *if and only if* $D^\alpha \neq (0)$ *and* $D^\beta \neq (0)$*; in addition,* $\dim D^\mu = \binom{n}{c}(\dim D^\alpha)(\dim D^\beta)$.

(iii) *Suppose that* $D^\mu \neq (0)$. *Then*

$$d_{\lambda\mu} = \begin{cases} d_{\sigma\alpha} d_{\tau\beta}, & \text{if } b = c, \\ 0, & \text{if } b \neq c. \end{cases}$$

Proof For part (i) we don't actually need to assume that R is a field. By definition, $S^\sigma \otimes S^\tau = (z_\sigma \otimes z_\tau)\mathscr{H}_b \otimes \mathscr{H}_{n-b}$ where $z_\sigma \otimes z_\tau = m_\sigma \otimes m_\tau + \overline{N}^\sigma \otimes \overline{N}^\tau$. By Corollary 4.9 and Lemma 4.6, $\hat{\mathrm{H}}_{s,b}(\overline{N}^\sigma \otimes \overline{N}^\tau) \cong \Theta_b(\overline{N}^\sigma \otimes \overline{N}^\tau)V^b \cong V^b \overline{N}^\lambda$; therefore, $\hat{\mathrm{H}}_{s,b}(S^\sigma \otimes S^\tau) \cong S^\lambda$ by the remarks following Corollary 4.9. (Notice that in this case the formula $\dim S^\lambda = \binom{n}{b}\dim(S^\sigma \otimes S^\tau)$ is just the combinatorial identity $|\operatorname{Std}(\lambda)| = \binom{n}{b}|\operatorname{Std}(\sigma)| \cdot |\operatorname{Std}(\tau)|$.)

For (ii) note that because $\hat{\mathrm{H}}_{s,c}$ is an equivalence of categories it takes simple $\mathscr{H}_b \otimes \mathscr{H}_{n-b}$–modules to simple $\mathscr{H}$–modules. Hence, $D^\mu \cong \hat{\mathrm{H}}_{s,c}(D^\alpha \otimes D^\beta)$ by induction on the dominance ordering using part (i) and (2.6); the dimension formula now follows from Corollary 4.9.

Finally, as $\hat{\mathrm{H}}_s$ takes composition series to composition series, part (iii) follows from (i) and (ii) and the decomposition $\mathbf{Mod}_{\mathscr{H}} = \bigoplus_{a=0}^n \mathbf{Mod}_{\mathscr{H}(a)}$. □

Note that part (iii) of the Proposition provides a recipe for calculating the decomposition matrix of $\mathscr{H}$ from the decomposition matrices of the "smaller" Ariki–Koike algebras $\mathscr{H}_{q,\mathbf{Q}_1}(b)$ and $\mathscr{H}_{q,\mathbf{Q}_2}(n-b)$ for $0 \leq b \leq n$.

Proposition 4.11 and all of the other consequences of Theorem 1.6 can be extended to the general case of Theorem 1.1, where the parameter set $\mathbf{Q}$ is partitioned into an arbitrary number of pieces. The notation needed to describe this is rather cumbersome so we leave the details to the reader.

5 The cyclotomic q–Schur algebra

We now extend the Morita equivalence of the previous section to cyclotomic q–Schur algebras. Let $\Gamma \subseteq \Lambda$ be a finite set of multicompositions of n with the property that whenever $\mu \in \Lambda^+$ and $\mu \trianglerighteq \lambda$ for some $\lambda \in \Gamma$ then $\mu \in \Gamma$. The cyclotomic q–Schur algebra associated with Γ is the algebra

$$\mathscr{S}_{q,\mathbf{Q}}(\Gamma) = \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\lambda \in \Gamma} M^\lambda\Big).$$

For the statement of Theorem 1.5 we set $\mathscr{S}_{q,\mathbf{Q}}(n) = \mathscr{S}_{q,\mathbf{Q}}(\Lambda^+)$.

Let $\Gamma^+ = \Gamma \cap \Lambda^+$ be the set of multipartitions in Γ. Then, by [**16**, Theorem 6.6] (see also (2.11)), the algebra $\mathscr{S}_{q,\mathbf{Q}}(\Gamma)$ has (cellular) basis

$$\{\, \varphi_{\mathtt{ST}} \mid \mathtt{S} \in \mathcal{T}_0(\lambda,\mu) \text{ and } \mathtt{T} \in \mathcal{T}_0(\lambda,\nu) \text{ for some } \mu,\nu \in \Gamma \text{and some} \lambda \in \Gamma^+ \,\},$$

where $\varphi_{\mathtt{ST}}$ is the $\mathscr{H}$–module homomorphism given by $\varphi_{\mathtt{ST}}(m_\alpha h) = \delta_{\alpha\nu} m_{\mathtt{ST}} h$. The basis $\{\varphi_{\mathtt{ST}}\}$ is the semistandard basis of $\mathscr{S}_{q,\mathbf{Q}}(\Gamma)$.

As before, we fix an integer s, with $1 \le s \le r$, such that $f_s(q,\mathbf{Q})$ is invertible in R and let $\Gamma_b = \{\,(\lambda^{(1)},\dots,\lambda^{(r)}) \in \Gamma \mid b = |\lambda^{(1)}| + \dots + |\lambda^{(s)}|\,\}$. We need to define analogues of the sets $\Lambda^+(b\colon s)$; however, we must be a little careful. Let

$$\Gamma_l(b\colon s) = \{\,\sigma \in \Lambda(b\colon s) \mid (\sigma,\tau) \in \Gamma \text{ for some } \tau \in \Lambda(n-b\colon r-s)\,\}$$

and

$$\Gamma_r(n-b\colon r-s) = \{\,\tau \in \Lambda(n-b\colon r-s) \mid (\sigma,\tau) \in \Gamma \text{ for some } \sigma \in \Lambda(b\colon s)\,\}.$$

Also let $\Gamma_l^+(b\colon s) = \Gamma_l(b\colon s) \cap \Lambda^+(b\colon s)$ and $\Gamma_r^+(n-b\colon r-s) = \Gamma_r(n-b\colon r-s) \cap \Lambda^+(n-b\colon r-s)$. Consider $\Gamma_l(b\colon s) \times \Gamma_r(n-b\colon r-s)$ as a poset in the obvious way.

5.1 Lemma *Suppose that $0 \le b \le n$. Then $\Gamma_l(b\colon s) \times \Gamma_r(n-b\colon r-s)$ and Γ_b are naturally isomorphic posets. In particular, if $(\alpha,\beta) \in \Lambda^+(b\colon s) \times \Lambda^+(n-b\colon r-s)$ and $(\alpha,\beta) \trianglerighteq (\sigma,\tau)$ for some $(\sigma,\tau) \in \Gamma_l(b\colon s) \times \Gamma_r(n-b\colon r-s)$ then $(\alpha,\beta) \in \Gamma_l(b\colon s) \times \Gamma_r(n-b\colon r-s)$.*

Proof The isomorphism is given by

$$\big((\sigma^{(1)},\dots,\sigma^{(s)}),(\tau^{(1)},\dots,\tau^{(r-s)})\big) \longmapsto (\sigma^{(1)},\dots,\sigma^{(s)},\tau^{(1)},\dots,\tau^{(r-s)}).$$

This is a poset isomorphism because $|\sigma^{(1)}| + \dots + |\sigma^{(s)}| = b$ whenever (σ,τ) is an element of $\Gamma_l(b\colon s) \times \Gamma_r(n-b\colon r-s)$. □

Let $\mathscr{S}_b(\Gamma) \otimes \mathscr{S}_{n-b}(\Gamma) = \mathscr{S}_{q,\mathbf{Q}_1}\big(\Gamma_l(b\colon s)\big) \otimes \mathscr{S}_{q,\mathbf{Q}_2}\big(\Gamma_r(n-b\colon r-s)\big)$, where $\mathbf{Q}_1 = (Q_1,\dots,Q_s)$ and $\mathbf{Q}_2 = (Q_{s+1},\dots,Q_r)$. The point of the Lemma is that it allows us to index the representations of $\mathscr{S}_b(\Gamma) \otimes \mathscr{S}_{n-b}(\Gamma)$ with the elements of Γ_b. For convenience we identity Γ_b and $\Gamma_l(b\colon s) \times \Gamma_r(n-b\colon r-s)$ in the sequel.

We can now give the analogue of Theorem 1.6 for a cyclotomic q–Schur algebra. Rather than introduce the notation necessary to state the general result, we consider only the special case where $\mathbf{Q}$ is partitioned into two pieces. See Theorem 1.5 for a special case of the more general result.

5.2 Theorem *Let $1 \le s \le r$ and suppose that $f_s(q,\mathbf{Q})$ is an invertible element of R. Then the cyclotomic q–Schur algebra $\mathscr{S}_{q,\mathbf{Q}}(\Gamma)$ is Morita equivalent to*

$$\mathscr{S}_s(\Gamma) = \bigoplus_{b=0}^{n} \mathscr{S}_b(\Gamma) \otimes \mathscr{S}_{n-b}(\Gamma).$$

Before we can give the proof we require some preparation. The basic idea is to investigate how the functor H_s acts on $\bigoplus_{\lambda\in\Gamma} M^\lambda$; in fact, it is easier to work with $\hat{\mathrm{H}}_s$.

5.3 Lemma *Suppose that $0 \le b \le n$ and let $\lambda = (\sigma,\tau) \in \Gamma_b$. Then*

$$\hat{\mathrm{H}}_{s,b}(M^\sigma \otimes M^\tau) = \theta_b(M^\lambda).$$

Proof We first note that $u_b^+ = m_{\omega_b}$ is a left factor of u_λ^+ because $\lambda \in \Lambda_b^+$; consequently, M^λ is a submodule of M^{ω_b} and so $\theta_b(M^\lambda)$ makes sense. Now $\Theta_b(m_\sigma \otimes m_\tau)v_b = \theta_b(m_\lambda)$ by Lemma 4.6; therefore, $\hat{\mathrm{H}}_{s,b}(M^\sigma \otimes M^\tau) = \theta_b(m_\lambda)\mathscr{H} = \theta_b(M^\lambda)$ by Corollary 4.9. □

Combining Theorem 3.16 and Corollary 3.18 we see that θ_b projects M^{ω_b} onto V^b and consequently that V^b is the unique direct summand of M^{ω_b} which belongs to $\mathbf{Mod}_{\mathscr{H}(b)}$. Now, $\mathscr{H} = \bigoplus_{c=0}^n \mathscr{H}(c)$ so we can write $M^\lambda = \bigoplus_{c=0}^n M^\lambda(c)$, where $M^\lambda(c)$ is the largest direct summand of M^λ which is contained in $\mathscr{H}(c)$. Furthermore, since M^λ is a quotient of M^{ω_b}, $M^\lambda(c) = (0)$ whenever $c < b$ by Corollary 3.18. To proceed we need to understand the direct summands $M^\lambda(c)$ of M^λ.

Given a multicomposition μ let $\overline{\mu} = (\overline{\mu}^{(1)}, \dots, \overline{\mu}^{(r)})$ be the (unique) multipartition where $\overline{\mu}^{(c)}$ is obtained by ordering the parts of $\mu^{(c)}$, for $1 \le c \le r$.

5.4 [**25**, Corollary 3.5] *There exists a family $\{\, Y^\lambda \mid \lambda \in \Lambda^+ \,\}$ of indecomposable $\mathscr{H}$–modules which are uniquely determined by the property that, for each $\mu \in \Lambda$,*

$$M^\mu \cong Y^{\overline{\mu}} \oplus \bigoplus_{\substack{\lambda\in\Lambda^+\\ \lambda\rhd\overline{\mu}}} c_{\lambda\overline{\mu}} Y^\lambda$$

for some non–negative integers $c_{\lambda\overline{\mu}}$ (which depend only on λ and $\overline{\mu}$). Moreover, S^λ is a quotient of Y^λ.

The Y^λ are generalizations of the Young modules of the symmetric groups. For the case of the q–Schur algebra (that is, when $r = 1$) this result is proved in [**13**].

Set $c_{\lambda\lambda} = 1$ and $c_{\lambda\overline{\mu}} = 0$ if $\lambda \not\trianglerighteq \overline{\mu}$. We can now identify the summand of M^μ which is contained in $\mathscr{H}(b)$.

5.5 Corollary *Suppose that $\mu \in \Gamma$. Then $M^\mu(b) \cong \bigoplus_{\lambda\in\Gamma_b^+} c_{\lambda\overline{\mu}} Y^\lambda$ for $0 \le b \le n$.*

Proof If $\lambda \in \Lambda_b^+$ then $S^\lambda \in \mathbf{Mod}_{\mathscr{H}(b)}$ by Lemma 4.10; therefore, Y^λ also belongs to $\mathbf{Mod}_{\mathscr{H}(b)}$, since Y^λ is indecomposable and S^λ is a quotient of Y^λ. (Observe that $\overline{\mu} \trianglerighteq \mu$; consequently, if $\lambda \in \Lambda^+$ and $\lambda \trianglerighteq \overline{\mu}$ then $\lambda \trianglerighteq \mu$ and so $\lambda \in \Gamma^+$.) As the decomposition of M^μ into a direct sum of indecomposables is unique up to isomorphism, the result follows. □

Let $M_\Gamma = \bigoplus_{\mu\in\Gamma} M^\mu$ and $M_{\Gamma_b} = \bigoplus_{(\sigma,\tau)\in\Gamma_b} M^\sigma \otimes M^\tau$. Then, by definition, $\mathscr{S}_{q,\mathbf{Q}}(\Gamma) = \mathrm{End}_{\mathscr{H}}(M_\Gamma)$ and $\mathscr{S}_s(\Gamma) = \bigoplus_{b=0}^n \mathrm{End}_{\mathscr{H}_b\otimes\mathscr{H}_{n-b}}(M_{\Gamma_b})$. Finally, we set $M_{\Gamma,s} = \bigoplus_{b=0}^n \hat{\mathrm{H}}_{s,b}(M_{\Gamma_b})$. These modules are the analogues of q–tensor space for the various cyclotomic q–Schur algebras; compare with [**14**].

5.6 Lemma *Suppose that $f_s(q,\mathbf{Q})$ is invertible. Then $M_{\Gamma,s}$ and M_Γ have the same set of indecomposable direct summands.*

Proof Applying the definitions,

$$M_{\Gamma,s} = \bigoplus_{b=0}^{n} \hat{\mathrm{H}}_{\mathfrak{s},b}(M_{\Gamma_b}) \cong \bigoplus_{b=0}^{n} \bigoplus_{(\sigma,\tau)\in\Gamma_b} \hat{\mathrm{H}}_{\mathfrak{s},b}(M^\sigma \otimes M^\tau).$$

Now, if $\mu = (\sigma,\tau) \in \Gamma_b$ then $\hat{\mathrm{H}}_{\mathfrak{s},b}(M^\sigma \otimes M^\tau) = M^\mu(b)$ by Lemma 5.3; on the other hand, $M^\mu(b) \cong \bigoplus_{\lambda\in\Gamma_b^+} c_{\lambda\overline{\mu}} Y^\lambda$ by Corollary 5.5. Therefore,

$$M_{\Gamma,s} \cong \bigoplus_{b=0}^{n} \bigoplus_{\mu\in\Gamma_b} \bigoplus_{\lambda\in\Gamma_b^+} c_{\lambda\overline{\mu}} Y^\lambda.$$

Consequently, $\{\, Y^\lambda \mid \lambda \in \Gamma^+ \,\}$ is a complete set of isomorphism classes of indecomposable direct summands of $M_{\Gamma,s}$. By (5.4) this is also the set of indecomposable direct summands of M_Γ, so this proves the Lemma. □

Proof of Theorem 5.2 By Lemma 5.6, the sets of indecomposable direct summands of the $\mathscr{H}$–modules M_Γ and $M_{\Gamma,s}$ coincide, except that the multiplicities of each Y^λ in the two modules will typically differ. Therefore, by a general argument, the $\mathscr{H}$–module endomorphism rings of M_Γ and $M_{\Gamma,s}$ are Morita equivalent. On the other hand, since $\hat{\mathrm{H}}_s$ is an equivalence of categories, it induces an isomorphism of endomorphism rings; so,

$$\mathscr{S}_s(\Gamma) = \bigoplus_{b=0}^{n} \mathrm{End}_{\mathscr{H}_b\otimes\mathscr{H}_{n-b}}(M_{\Gamma_b}) \cong \bigoplus_{b=0}^{n} \mathrm{End}_{\mathscr{H}}\big(\hat{\mathrm{H}}_{s,b}(M_{\Gamma_b})\big) \cong \mathrm{End}_{\mathscr{H}}\big(M_{\Gamma,s}\big),$$

where the last isomorphism is a consequence of the decomposition $\mathbf{Mod}_{\mathscr{H}} = \bigoplus_{b=0}^{n} \mathbf{Mod}_{\mathscr{H}(b)}$. As we have already observed that $\mathrm{End}_{\mathscr{H}}\big(M_{\Gamma,s}\big)$ is Morita equivalent to $\mathscr{S}_{q,\mathbf{Q}}(\Gamma) = \mathrm{End}_{\mathscr{H}}(M_\Gamma)$, the proof of Theorem 5.2 is complete.

□

To conclude, we note that Proposition 4.11 also generalizes to the cyclotomic q–Schur algebra case. Formally, the proof is similar to the Ariki–Koike case, so we omit the details.

Recall that the cell modules of $\mathscr{S}_{q,\mathbf{Q}}(\Gamma)$ are called Weyl modules and are denoted by W^λ, for $\lambda \in \Gamma^+$. As with the Specht modules, there is a symmetric associative bilinear form on W^λ; let $F^\lambda = W^\lambda/\operatorname{rad} W^\lambda$, where $\operatorname{rad} W^\lambda$ is the radical of this form. Then $\{\, F^\lambda \mid \lambda \in \Gamma^+ \,\}$ is a complete set of pairwise non–isomorphic irreducible $\mathscr{S}_{q,\mathbf{Q}}(\Gamma)$–modules by [**16**, Theorem 6.16]. Furthermore, by [**25**, Theorem 2.3] if $D^\mu \neq (0)$ then $[W^\lambda : F^\mu] = [S^\lambda : D^\mu]$; because of this we abuse notation and write $d_{\lambda\mu} = [W^\lambda : F^\mu]$ for all $\lambda,\mu \in \Gamma^+$.

Again, we write $\hat{\mathrm{H}}_s$ and $\hat{\mathrm{H}}_{s,b}$ for the induced functors between the categories $\mathbf{Mod}_{\mathscr{S}_s(\Gamma)}$ and $\mathbf{Mod}_{\mathscr{S}_{q,\mathbf{Q}}(\Gamma)}$ and their subcategories.

5.7 Corollary *Suppose that R is a field and that $f_s(q,\mathbf{Q}) \neq 0$. Suppose that $\lambda = (\sigma,\tau) \in \Gamma_b^+$ and $\mu = (\alpha,\beta) \in \Gamma_c^+$ where $0 \le b,c \le n$. Then we have the following.*

(i) $W^\lambda \cong \hat{\mathrm{H}}_s(W^\sigma \otimes W^\tau) = \hat{\mathrm{H}}_{s,b}(W^\sigma \otimes W^\tau)$.

(ii) $F^\mu \cong \hat{\mathrm{H}}_s(F^\alpha \otimes F^\beta) = \hat{\mathrm{H}}_{s,c}(F^\alpha \otimes F^\beta)$. *Consequently, $F^\mu \neq (0)$ if and only if $F^\alpha \neq (0)$ and $F^\beta \neq (0)$.*

(iii) *Finally, if $b = c$ then $d_{\lambda\mu} = d_{\sigma\alpha} d_{\tau\beta}$; otherwise, $d_{\lambda\mu} = 0$.*

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